

The shifted fourth moment of automorphic L -functions of prime power level

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Abstract

We prove the asymptotic formula for the fourth moment of automorphic L -functions of level p^ν , where p is a fixed prime number and $\nu \rightarrow \infty$. This paper is a continuation of work by Rouymi, who computed asymptotics of the first three moments at prime power level, and a generalization of results obtained for prime level by Duke, Friedlander & Iwaniec and Kowalski, Michel & Vanderkam.

1 Introduction

Let $L(s, f)$ be an automorphic L -function associated to $f \in H_k^*(q)$, where $H_k^*(q)$ denote the set of primitive forms of weight k and level q . An important subject in analytic number theory is the behavior of such L -functions near the critical line. Questions of particular interest are subconvexity bounds and proportion of non-vanishing L -values. A possible way to analyze these problems is the method of moments. This technique proved to be very effective in the recent years; see [D], [DFI], [IS], [KM], [KMV] for details and examples.

In 1995 Duke [D] proved the asymptotic formula for the first moment and the upper bound for the second moment when q is prime and $k = 2$. Four years later, Akbary [A] generalized this result to the case of prime

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q and $k > 2$. In 2011 Ichihara [Ich] found the asymptotic formula for the first moment when q is a power of prime and $2 \leq k \leq 10$, $k = 14$. In the same year, Rouymi [R] computed asymptotics of the first, second and third moments when q is a power of prime and k is an arbitrary fixed even integer.

In order to break the convexity barrier, one needs to evaluate the limit moment of order

$$(1.1) \quad \kappa_0 := \liminf_{q \rightarrow \infty} 4 \frac{\log |H_k^*(q)|}{\log q} = 4.$$

At the same time, starting from the κ_0 -th moment the main term of asymptotic formula contains nontrivial non-diagonal contribution. See [M] for details.

The moment of order $\kappa_0 = 4$ for prime q , $q \rightarrow \infty$ was studied by Duke, Friedlander & Iwaniec [DFI] and Kowalski, Michel & Vanderkam [KMV]. The main term of the fourth moment splits into diagonal M^D , off-diagonal M^{OD} and off-off-diagonal M^{OOD} parts. Therefore, it requires three different stages of analysis.

Theorem 1.1. (*[KMV], corollary 1.3*) *Let q be a prime number and $k = 2$. For all $\epsilon > 0$*

$$(1.2) \quad \sum_{f \in H_k^*(q)}^h L(1/2, f)^4 = R(\log q) + O_\epsilon(q^{-1/12+\epsilon}),$$

where R is a polynomial of degree 6 and the leading coefficient is $\frac{1}{60\pi^2}$.

In this paper, the result of theorem 1.1 is extended as follows.

- We consider the level of the form $q = p^\nu$, where p is a fixed prime number and $\nu \rightarrow \infty$.
- We assume that the weight $k \geq 2$ is an arbitrary fixed even integer.
- We slightly shift each L -function in the product from the critical line $\Re s = 1/2$

$$M_4(\mathbf{t}, \mathbf{r}) = \sum_{f \in H_k^*(q)}^h |L(1/2 + t_1 + ir_1, f)|^2 |L(1/2 + t_2 + ir_2, f)|^2,$$

where $\mathbf{t} = (t_1, t_2)$, $\mathbf{r} = (r_1, r_2)$, $t_1, t_2, r_1, r_2 \in \mathbb{R}$ and $|t_1|, |t_2| < \frac{1}{\log q}$.

The shifts simplify analysis of the off-off-diagonal term, reveal more clearly a combinatorial structure of mean values and allow us to verify random matrix theory conjectures including all lower order terms by Conrey, Farmer, Keating, Rubinstein and Snaith [CFKRS].

We introduce the following notations

$$(1.3) \quad \hat{q} = \frac{\sqrt{q}}{2\pi},$$

$$(1.4) \quad \zeta_q(s) = \zeta(s)(1 - p^{-s}).$$

Conjecture 1.2. (*analog of conjectures 4.5.1 and 4.5.2 in [CFKRS]*) Let $q = p^\nu$, where p is a fixed prime and $\nu \geq 3$. Let $k > 0$ be an even integer. Up to an error term, we have

$$\begin{aligned} M_4(\mathbf{t}, \mathbf{r}) &= \frac{\phi(q)}{q} \hat{q}^{-2t_1-2t_2} \sum_{\substack{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 = \pm 1 \\ \epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4 = 1}} \hat{q}^{t_1(\epsilon_1 + \epsilon_2) + t_2(\epsilon_3 + \epsilon_4) + ir_1(\epsilon_1 - \epsilon_2) + ir_2(\epsilon_3 - \epsilon_4)} \\ &\quad \times \left(\frac{\Gamma(-t_1 - ir_1 + k/2) \Gamma(-t_1 + ir_1 + k/2) \Gamma(-t_2 - ir_2 + k/2)}{\Gamma(t_1 + ir_1 + k/2) \Gamma(t_1 - ir_1 + k/2) \Gamma(t_2 + ir_2 + k/2)} \right)^{1/2} \\ &\quad \times \left(\frac{\Gamma(-t_2 + ir_2 + k/2) \Gamma(\epsilon_1(t_1 + ir_1) + k/2) \Gamma(\epsilon_2(t_1 - ir_1) + k/2)}{\Gamma(t_2 - ir_2 + k/2) \Gamma(-\epsilon_1(t_1 + ir_1) + k/2) \Gamma(-\epsilon_2(t_1 - ir_1) + k/2)} \right)^{1/2} \\ &\quad \times \left(\frac{\Gamma(\epsilon_3(t_2 + ir_2) + k/2) \Gamma(\epsilon_4(t_2 - ir_2) + k/2)}{\Gamma(-\epsilon_3(t_2 + ir_2) + k/2) \Gamma(-\epsilon_4(t_2 - ir_2) + k/2)} \right)^{1/2} \\ &\quad \times \frac{\zeta_q(1 + t_1(\epsilon_1 + \epsilon_2) + ir_1(\epsilon_1 - \epsilon_2)) \zeta_q(1 + t_2(\epsilon_3 + \epsilon_4) + ir_2(\epsilon_3 - \epsilon_4))}{\zeta_q(2 + t_1(\epsilon_1 + \epsilon_2) + t_2(\epsilon_3 + \epsilon_4) + ir_1(\epsilon_1 - \epsilon_2) + ir_2(\epsilon_3 - \epsilon_4))} \\ &\quad \times \zeta_q(1 + \epsilon_1(t_1 + ir_1) + \epsilon_3(t_2 + ir_2)) \zeta_q(1 + \epsilon_1(t_1 + ir_1) + \epsilon_4(t_2 - ir_2)) \\ &\quad \times \zeta_q(1 + \epsilon_2(t_1 - ir_1) + \epsilon_3(t_2 + ir_2)) \zeta_q(1 + \epsilon_2(t_1 - ir_1) + \epsilon_4(t_2 - ir_2)). \end{aligned}$$

Main Theorem 1.3. Let $\theta = 7/64$, $q = p^\nu$, where p is a fixed prime and $\nu \geq 3$. Let $k > 0$ be an even integer. For all $\epsilon > 0$ the fourth moment can be decomposed as follows

$$M_4(\mathbf{t}, \mathbf{r}) = M^D + M^{OD} + M^{OOD} + O_{\epsilon, p, k} \left(q^\epsilon \left(q^{-\frac{k-1-2\theta}{8-8\theta}} + q^{-1/4} \right) \right),$$

where the implied constant depends polynomially on r_1, r_2 . Furthermore,

$$\begin{aligned} (1.5) \quad M^D + M^{OD} &= \frac{\phi(q)}{q} \sum_{\epsilon_1, \epsilon_2 = \pm 1} \hat{q}^{-2t_1-2t_2+2\epsilon_1 t_1+2\epsilon_2 t_2} \frac{\Gamma(\epsilon_1 t_1 + ir_1 + k/2)}{\Gamma(t_1 + ir_1 + k/2)} \\ &\quad \times \frac{\Gamma(\epsilon_1 t_1 - ir_1 + k/2)}{\Gamma(t_1 - ir_1 + k/2)} \frac{\Gamma(\epsilon_2 t_2 + ir_2 + k/2) \Gamma(\epsilon_2 t_2 - ir_2 + k/2)}{\Gamma(t_2 + ir_2 + k/2) \Gamma(t_2 - ir_2 + k/2)} \\ &\quad \times \zeta_q(1 + 2\epsilon_1 t_1) \zeta_q(1 + 2\epsilon_2 t_2) \frac{\prod_{\epsilon_3, \epsilon_4 = \pm 1} \zeta_q(1 + \epsilon_1 t_1 + \epsilon_2 t_2 + i\epsilon_3 r_1 + i\epsilon_4 r_2)}{\zeta_q(2 + 2\epsilon_1 t_1 + 2\epsilon_2 t_2)} \end{aligned}$$

and

$$(1.6) \quad M^{OOD} = \frac{\phi(q)}{q} \sum_{\epsilon_1, \epsilon_2 = \pm 1} \hat{q}^{-2t_1 - 2t_2 + 2i\epsilon_1 r_1 + 2i\epsilon_2 r_2} \\ \times \frac{\Gamma(k/2 - t_1 + i\epsilon_1 r_1) \Gamma(k/2 - t_2 + i\epsilon_2 r_2)}{\Gamma(k/2 + t_1 - i\epsilon_1 r_1) \Gamma(k/2 + t_2 - i\epsilon_2 r_2)} \\ \times \zeta_q(1 + 2i\epsilon_1 r_1) \zeta_q(1 + 2i\epsilon_2 r_2) \frac{\prod_{\epsilon_3, \epsilon_4 = \pm 1} \zeta_q(1 + \epsilon_3 t_1 + \epsilon_4 t_2 + i\epsilon_1 r_1 + i\epsilon_2 r_2)}{\zeta_q(2 + 2i\epsilon_1 r_1 + 2i\epsilon_2 r_2)}.$$

Remark 1.4. The condition $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 = \pm 1$, $\epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4 = 1$ in conjecture 1.2 implies that there are eight terms in the sum. The four of them

$$(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) = (1, 1, 1, 1), (1, 1, -1, -1), (-1, -1, 1, 1), (-1, -1, -1, -1)$$

coincide with the summands of (1.5), and the other four

$$(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) = (-1, 1, -1, 1), (-1, 1, 1, -1), (1, -1, -1, 1), (1, -1, 1, -1)$$

with the summands of (1.6).

By letting the shifts tend to zero in theorem 1.3, we obtain the asymptotic formula for the fourth moment at the critical point $s = 1/2$.

Corollary 1.5. *Let $\theta = 7/64$, $q = p^\nu$, where p is a fixed prime and $\nu \geq 3$. Let $k > 0$ be an even integer. For all $\epsilon > 0$*

$$(1.7) \quad M_4(\mathbf{0}, \mathbf{0}) = \sum_{f \in H_k^*(q)}^h L(1/2, f)^4 \\ = R(\log q) + O_{\epsilon, k, p} \left(q^\epsilon \left(q^{-\frac{k-1-2\theta}{8-8\theta}} + q^{-1/4} \right) \right),$$

where R is a polynomial of degree 6 and the leading coefficient is

$$(1.8) \quad \left(\frac{\phi(q)}{q} \right)^7 \frac{p^2}{p^2 - 1} \frac{1}{60\pi^2}.$$

The paper is organized as follows. In section 2 we remind the reader of some definitions and fundamental results. Section 3 provides the explicit formula for the diagonal, off-diagonal and off-off-diagonal main terms. Asymptotics of the diagonal and off-diagonal terms is derived in section 4. Sections 5 and 6 are devoted to proving the asymptotic formula for the off-off-diagonal term. Corollary 1.5 is proved as a limit case at the end of sections 4 and 6.

2 Background information

The purpose of this section is to recall some results on automorphic forms and related subjects.

2.1 Automorphic L -functions

A holomorphic function f on the Poincaré upper-half plane $\mathbb{H} = \{z \in \mathbb{C}, \Im z > 0\}$ is called a *cusp form* of weight k and level q if it satisfies the following conditions:

$$(2.1) \quad f(\gamma z) = (cz + d)^k f(z)$$

$$\text{for all } \gamma \in \Gamma_0(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \text{ such that } c \equiv 0 \pmod{q} \right\},$$

$$(2.2) \quad (\Im z)^{k/2} |f(z)| \text{ is bounded on } \mathbb{H}.$$

Let $S_k(q)$ be the space of cusp forms of weight $k \geq 2$ and level q . It is equipped with the Petersson inner product

$$(2.3) \quad \langle f, g \rangle_q := \int_{F_0(q)} f(z) \overline{g(z)} y^k \frac{dx dy}{y^2},$$

where $F_0(q)$ is a fundamental domain of the action of $\Gamma_0(q)$ on \mathbb{H} . Any $f \in S_k(q)$ has a Fourier expansion at infinity

$$(2.4) \quad f(z) = \sum_{n \geq 1} a_f(n) e(nz).$$

According to the Atkin-Lehner theory [AL], the space $S_k(q)$ can be decomposed into two subspaces

$$(2.5) \quad S_k(q) = S_k^{new}(q) \oplus S_k^{old}(q).$$

The space of old forms contains cusp forms of level q coming from lower levels

$$(2.6) \quad S_k^{old}(q) = \text{span} \{ f(lz) : lq' | q, q' < q, f(z) \in S_k(q') \},$$

and the space of new forms is the orthogonal complement to $S_k^{old}(q)$.

We let $H_k(q)$ denote an orthogonal basis of the space of cusp forms $S_k(q)$ and $H_k^*(q)$ - an orthogonal basis of $S_k^{new}(q)$. Elements of $H_k^*(q)$ with normalized Fourier coefficients

$$(2.7) \quad \lambda_f(n) := a_f(n) n^{-(k-1)/2},$$

$$(2.8) \quad \lambda_f(1) = 1$$

are called *primitive forms*. Accordingly,

$$(2.9) \quad \lambda_f(n) \in \mathbb{R},$$

$$(2.10) \quad \lambda_f(n_1)\lambda_f(n_2) = \sum_{\substack{d|(n_1, n_2) \\ (d, q)=1}} \lambda_f\left(\frac{n_1 n_2}{d^2}\right).$$

Let $\operatorname{Re}(s) > 1$. Then for $f \in H_k^*(q)$ we define the *automorphic L-function* as

$$(2.11) \quad L(s, f) = \sum_{n \geq 1} \lambda_f(n) n^{-s}.$$

The *completed L-function*

$$(2.12) \quad \Lambda(s, f) = \left(\frac{\sqrt{q}}{2\pi}\right)^s \Gamma\left(s + \frac{k-1}{2}\right) L(s, f)$$

can be analytically continued on the whole complex plane and satisfies the functional equation

$$(2.13) \quad \Lambda(s, f) = \epsilon_f \Lambda(1-s, f),$$

where $s \in \mathbb{C}$ and $\epsilon_f = \pm 1$. We define the *harmonic average* over the set of primitive newforms by

$$(2.14) \quad \sum_{f \in H_k^*(q)}^h A(f) := \sum_{f \in H_k^*(q)} \frac{\Gamma(k-1)}{(4\pi)^{k-1} \langle f, f \rangle_q} A(f).$$

2.2 Kloosterman sums

Consider the sum

$$(2.15) \quad S(m, n, c) = \sum_{\substack{d \pmod{c} \\ (c, d)=1}} e\left(\frac{m\bar{d} + nd}{c}\right),$$

where $d\bar{d} \equiv 1 \pmod{c}$, $e(z) = \exp(2\pi i z)$.

The value of $S(m, n, c)$ is always a real number because

$$(2.16) \quad \overline{S(m, n, c)} = S(m, n, c).$$

Further,

$$(2.17) \quad S(m, n, c) = S(n, m, c),$$

$$(2.18) \quad S(ma, n, c) = S(m, na, c) \text{ if } (a, c) = 1.$$

Another important property is the twisted multiplicity ([Iw] formula (4.12)).

Suppose $(c_1, c_2) = 1$, $c_2 \overline{c_2} \equiv 1 \pmod{c_1}$, $c_1 \overline{c_1} \equiv 1 \pmod{c_2}$. Then

$$(2.19) \quad S(m, n, c_1 c_2) = S(m \overline{c_2}, n \overline{c_2}, c_1) S(m \overline{c_1}, n \overline{c_1}, c_2).$$

Lemma 2.1. (*Weil's bound, [We]*) One has

$$(2.20) \quad |S(m, n, c)| \leq (m, n, c)^{1/2} c^{1/2} \tau(c).$$

Lemma 2.2. (*Royer, [Ro], lemma A.12*) Let m, n, c be three strictly positive integers and p be a prime number. Suppose $p^2 | c$, $p | m$ and $p \nmid n$. Then $S(m, n, c) = 0$.

2.3 Large sieve inequality

Theorem 2.3. (*Deshouillers, Iwaniec, theorem 9 of [DI]*) Let r and s be positive coprime integers, C, M, N be positive real numbers and g be real-valued function of \mathbf{C}^6 class (first and second derivatives are continuous for each of variables) with support in $[M, 2M] \times [N, 2N] \times [C, 2C]$ such that

$$(2.21) \quad \left| \frac{\partial^{(j+k+l)}}{\partial m^{(j)} \partial n^{(k)} \partial c^{(l)}} g(m, n, c) \right| \leq M^{-j} N^{-k} C^{-l} \text{ for } 0 \leq j, k, l \leq 2.$$

Then for any $\epsilon > 0$ and complex sequences $\mathbf{a} = \{a_m\}$, $\mathbf{b} = \{b_n\}$ one has

$$(2.22) \quad \sum_{(c,r)=1} \sum_m a_m \sum_n b_n g(m, n, c) S(m\bar{r}, \pm n, sc) \ll_{\epsilon} \left(\sum_{M < m \leq 2M} |a_m|^2 \right)^{1/2} \left(\sum_{N < n \leq 2N} |b_n|^2 \right)^{1/2} C^{\epsilon} \left(1 + \frac{s\sqrt{r}C}{\sqrt{MN}} \right)^{2\theta} \\ \times \frac{(s\sqrt{r}C + \sqrt{MN} + \sqrt{sMC})(s\sqrt{r}C + \sqrt{MN} + \sqrt{sNC})}{s\sqrt{r}C + \sqrt{MN}}.$$

Here

$$\theta = \theta_{rs} := \sqrt{\max(0, 1/4 - \lambda_1)}$$

and $\lambda_1 = \lambda_1(rs)$ is the smallest positive eigenvalue for the Hecke congruence subgroup $\Gamma_0(rs)$. Currently the best known bound on λ_1 is due to Kim and Sarnak [KS]. Accordingly, we can take $\theta = 7/64$.

2.4 Petersson's trace formula

The key ingredient of our proof is the Petersson trace formula. It allows expressing average of Fourier coefficients of cusp forms in terms of Kloosterman sums weighted by J -Bessel functions.

Theorem 2.4. (*proposition 14.5, [IK]*) For $m, n \geq 1$ we have

$$(2.23) \quad \Delta_q(m, n) := \sum_{f \in H_k(q)}^h \lambda_f(m) \lambda_f(n) \\ = \delta_{m,n} + 2\pi i^{-k} \sum_{q|c} \frac{S(m, n, c)}{c} J_{k-1} \left(\frac{4\pi\sqrt{mn}}{c} \right).$$

If q is a prime number and $k < 12$, the Petersson trace formula also works for moments of L -functions associated to primitive forms because the space of old forms is empty.

When q is composite, one needs to exclude the contribution of old forms. Iwaniec, Luo and Sarnak constructed a special basis in order to find an analog of Petersson's trace formula for primitive forms of square-free level.

Theorem 2.5. (*proposition 2.8 of [ILS]*) Let q be square-free, $(m, q) = 1$ and $(n, q^2) | q$. Then

$$(2.24) \quad \Delta_q^*(m, n) := \sum_{f \in H_k^*(q)}^h \lambda_f(m) \lambda_f(n) \\ = \frac{k-1}{12} \sum_{LM=q} \frac{\mu(L)M}{(n, L) \prod_{p|(n, L)} (1+p^{-1})} \sum_{l|L^\infty} l^{-1} \Delta_M(ml^2, n).$$

This result was extended to the case of prime power level by Rouymi.

Theorem 2.6. (*remark 4 of [R]*) Let $q = p^\nu$, $\nu \geq 3$. Then

$$(2.25) \quad \Delta_q^*(m, n) := \sum_{f \in H_k^*(q)}^h \lambda_f(m) \lambda_f(n) \\ = \begin{cases} \Delta_q(m, n) - \frac{\Delta_{q/p}(m, n)}{p} & \text{if } (q, mn) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

2.5 Poisson type summation formula connected with the Eisenstein-Maass series

Let

$$(2.26) \quad \tau_v(n) = |n|^{v-1/2} \sigma_{1-2v}(n) = |n|^{v-1/2} \sum_{d|n, d>0} d^{1-2v}.$$

If $v = 1/2$, then $\tau_v(n)$ reduces to the divisor function $\tau(n)$. Furthermore, $\tau_v(n)$ satisfies the property of multiplicity (see [K], page 74)

$$(2.27) \quad \tau_v(n)\tau_v(m) = \sum_{d|(n,m)} \tau_v\left(\frac{nm}{d^2}\right).$$

Lemma 2.7. (*Ramanujan's identity, [T], page 8*)

Let $\Re s > 1 + |\Re v - 1/2| + |\Re \mu - 1/2|$. Then

$$(2.28) \quad \zeta(2s) \sum_{n \geq 1} \frac{\tau_v(n)\tau_\mu(n)}{n^s} = \zeta(s+v-\mu)\zeta(s-v+\mu)\zeta(s+v+\mu-1)\zeta(s-v-\mu+1).$$

If $v = \mu = 1/2$, this reduces to

$$(2.29) \quad \sum_{n \geq 1} \frac{\tau(n)^2}{n^s} = \frac{\zeta(s)^4}{\zeta(2s)}.$$

Consider the Bessel kernels expressed in terms of J and K -Bessel functions

$$(2.30) \quad k_0(x, v) := \frac{1}{2 \cos \pi v} (J_{2v-1}(x) - J_{1-2v}(x)),$$

$$(2.31) \quad k_1(x, v) := \frac{2}{\pi} \sin \pi v K_{2v-1}(x).$$

Theorem 2.8. (*theorem 5.2 of [K], page 89*) Let ϕ be a smooth, compactly supported function on \mathbb{R}^+ . Then for every v with $\Re v = 1/2$, $(c, d) = 1$, $c \geq 1$ one has

$$(2.32) \quad \frac{4\pi}{c} \sum_{m \geq 1} e\left(\frac{md}{c}\right) \tau_v(m) \phi\left(\frac{4\pi\sqrt{m}}{c}\right) = 2 \frac{\zeta(2v)}{(4\pi)^{2v}} \hat{\phi}(2v+1) + 2 \frac{\zeta(2-2v)}{(4\pi)^{2-2v}} \hat{\phi}(3-2v) + \sum_{m \geq 1} \tau_v(m) \int_0^\infty \left[e\left(-\frac{ma}{c}\right) k_0(x\sqrt{m}, v) + e\left(\frac{ma}{c}\right) k_1(x\sqrt{m}, v) \right] \phi(x) x dx,$$

where $ad \equiv 1 \pmod{c}$ and $\hat{\phi}$ is the Mellin transform of ϕ .

2.6 Quadratic divisor problem

Applying formula (2.32), we generalize theorem 1 of [DFI2] as follows.

Theorem 2.9. *Let $a, b \geq 1$, $(a, b) = 1$, $h \neq 0$, $r_1, r_2 \in \mathbb{R}$. Let*

$$D_f(a, b; h) = \sum_{am \mp bn = h} \tau_{1/2+ir_1}(m) \tau_{1/2+ir_2}(n) f(am, bn)$$

with

$$(2.33) \quad x^i y^j f^{(ij)}(x, y) \ll \left(1 + \frac{x}{X}\right)^{-1} \left(1 + \frac{y}{Y}\right)^{-1} Q^{i+j}.$$

Assume that

$$(2.34) \quad ab < Q^{-5/4} (X + Y)^{-5/4} (XY)^{1/4+\epsilon}.$$

Then

$$D_f(a, b; h) = \int_0^\infty g(x, \pm x \mp h) dx + O(Q^{5/4} (X + Y)^{1/4} (XY)^{1/4+\epsilon}),$$

where the implied constant depends polynomially on r_1, r_2 . Here $g(x, y) = f(x, y) \Lambda_{a,b,h}(x, y)$ with

$$(2.35) \quad \Lambda_{a,b,h}(x, y) := \sum_{w=1}^\infty S(0, h, w) \sum_{\epsilon_1, \epsilon_2 = \pm 1} \frac{(ab, w)(a, w)^{2i\epsilon_1 r_1} (b, w)^{2i\epsilon_2 r_2}}{a^{1+i\epsilon_1 r_1} b^{1+i\epsilon_2 r_2} w^{2+2i\epsilon_1 r_1+2i\epsilon_2 r_2}} \\ \times \zeta(1 + 2i\epsilon_1 r_1) \zeta(1 + 2i\epsilon_2 r_2) x^{i\epsilon_1 r_1} y^{i\epsilon_2 r_2}.$$

3 The fourth moment: preliminary steps

3.1 Approximate functional equation

Let $P_r(s)$ be an even polynomial vanishing at all poles of $\Gamma(s+ir+k/2)\Gamma(s-ir+k/2)$ in the range $\Re s \geq -L$ for some large constant $L > 0$. For $t, r \in \mathbb{R}$ we define

$$(3.1) \quad W_{t,r}(y) := \frac{1}{2\pi i} \int_{(3)} \frac{P_r(s)}{P_r(t)} \zeta_q(1+2s) \\ \times \frac{\Gamma(s+ir+k/2)\Gamma(s-ir+k/2)}{\Gamma(t+ir+k/2)\Gamma(t-ir+k/2)} y^{-s} \frac{2s ds}{s^2 - t^2}.$$

Lemma 3.1. *Suppose $y > 0$, $|t| < 1/2$. For any $C > |t|$*

$$(3.2) \quad W_{t,r}(y) = O_{C,t,r}(y^{-C}) \text{ as } y \rightarrow \infty,$$

$$(3.3) \quad W_{t,r}(y) = \zeta_q(1-2t)y^t \frac{\Gamma(-t+ir+k/2)\Gamma(-t-ir+k/2)}{\Gamma(t+ir+k/2)\Gamma(t-ir+k/2)} \\ + \zeta_q(1+2t)y^{-t} + O_{C,t,r}(y^C) \text{ as } y \rightarrow 0.$$

The implied constants depend polynomially on r .

Proof. Asymptotic expansion for the ratio of gamma functions gives

$$\frac{\Gamma(C+ir+k/2)\Gamma(C-ir+k/2)}{\Gamma(t+ir+k/2)\Gamma(t-ir+k/2)} = (|r|)^{2(C-t)} (1 + O(1/|r|)).$$

First, without crossing any poles, we can shift the contour of integration to $\Re s = C$ with $C > |t|$. This implies (3.2). Second, we move the contour of integration to $\Re s = -C$, meeting two simple poles at $s = \pm t$. Therefore, as $y \rightarrow 0$, we have (3.3). \square

Lemma 3.2. For $t, r \in \mathbb{R}$, $|t| < 1/2$ we have

$$(3.4) \quad |L(1/2+t+ir, f)|^2 = (\hat{q})^{-2t} \sum_{n \geq 1} \tau_{1/2+ir}(n) \frac{\lambda_f(n)}{\sqrt{n}} W_{t,r} \left(\frac{n}{\hat{q}^2} \right).$$

Proof. Consider

$$I_t := \frac{1}{2\pi i} \int_{(3)} \Lambda(1/2+s+ir, f) \Lambda(1/2+s-ir, f) \frac{P_r(s)}{s-t} ds.$$

Moving the contour of integration to $\Re s = -3$, we pick up a simple pole at $s = t$. The functional equation (2.13) implies that

$$I_t + \epsilon_f^2 I_{-t} = \text{Res}_{s=t} \left(\Lambda(1/2+s+ir, f) \Lambda(1/2+s-ir, f) \frac{P_r(s)}{s-t} \right) \\ = P_r(t) \Lambda(1/2+t+ir, f) \Lambda(1/2+t-ir, f).$$

Observe that for $s > 1/2$, the property 2.10 yields

$$|L(1/2+s+ir, f)|^2 = \zeta_q(1+2s) \sum_{n \geq 1} \frac{\lambda_f(n)}{n^{1/2+s}} \tau_{1/2+ir}(n).$$

Finally,

$$|L(1/2+t+ir, f)|^2 = (\hat{q})^{-2t} \sum_{n \geq 1} \tau_{1/2+ir}(n) \frac{\lambda_f(n)}{\sqrt{n}} \\ \times \frac{1}{2\pi i} \int_{(3)} \frac{P_r(s)}{P_r(t)} \zeta_q(1+2s) \frac{\Gamma(s+ir+k/2)\Gamma(s-ir+k/2)}{\Gamma(t+ir+k/2)\Gamma(t-ir+k/2)} \left(\frac{n}{\hat{q}^2} \right)^{-s} \frac{2s ds}{s^2 - t^2}.$$

\square

Corollary 3.3. *The fourth moment can be written as follows*

$$(3.5) \quad M_4(\mathbf{t}, \mathbf{r}) = (\hat{q})^{-2t_1-2t_2} \sum_{m,n \geq 1} \tau_{1/2+ir_1}(m) \tau_{1/2+ir_2}(n) \\ \times \frac{1}{\sqrt{mn}} W_{t_1, r_1} \left(\frac{m}{\hat{q}^2} \right) W_{t_2, r_2} \left(\frac{n}{\hat{q}^2} \right) \Delta_q^*(m, n).$$

3.2 Applying the Petersson trace formula

Here we apply theorem 2.6 for $\nu \geq 3$. The case $\nu = 2$ can be treated similarly, but doesn't seem to be of particular interest since the final goal is $\nu = \infty$. Let

$$(3.6) \quad T(c) := c \sum_{\substack{m,n \geq 1 \\ (q, mn)=1}} \frac{\tau_{1/2+ir_1}(m) \tau_{1/2+ir_2}(n)}{\sqrt{nm}} \\ \times W_{t_1, r_1} \left(\frac{m}{\hat{q}^2} \right) W_{t_2, r_2} \left(\frac{n}{\hat{q}^2} \right) S(m, n, c) J_{k-1} \left(\frac{4\pi\sqrt{mn}}{c} \right).$$

Using the trace formula (2.25), the fourth moment (3.5) can be written as a sum of diagonal and non-diagonal parts.

Proposition 3.4. *The following decomposition takes place*

$$(3.7) \quad M_4(\mathbf{t}, \mathbf{r}) = M^D + M_1^{ND} + M_2^{ND},$$

where

$$(3.8) \quad M^D = \frac{\phi(q)}{q} \hat{q}^{-2t_1-2t_2} \\ \times \sum_{\substack{n \geq 1 \\ (q, n)=1}} \frac{\tau_{1/2+ir_1}(n) \tau_{1/2+ir_2}(n)}{n} W_{t_1, r_1} \left(\frac{n}{\hat{q}^2} \right) W_{t_2, r_2} \left(\frac{n}{\hat{q}^2} \right),$$

$$(3.9) \quad M_1^{ND} = 2\pi i^{-k} \hat{q}^{-2t_1-2t_2} \sum_{q|c} \frac{1}{c^2} T(c),$$

and

$$(3.10) \quad M_2^{ND} = -\frac{2\pi i^{-k}}{p} \hat{q}^{-2t_1-2t_2} \sum_{\frac{q}{p}|c} \frac{1}{c^2} T(c).$$

Remark 3.5. For any $\epsilon > 0$ we have $M^D \ll_{\epsilon, \mathbf{r}} \frac{\phi(q)}{q} q^\epsilon$. The asymptotics of this term will be evaluated in section 4.2.

3.3 Smooth partition of unity and restriction of summations

Assume that $F_X(x)$ is a compactly supported function in $[X/2, 3X]$ such that for any integral $j \geq 0$

$$(3.11) \quad x^j F_X^{(j)}(x) \ll_j 1.$$

We make a smooth dyadic partition of unity (see Appendix A of [RR] for details). Accordingly,

$$\frac{1}{\sqrt{mn}} W_{t_1, r_1} \left(\frac{m}{\hat{q}^2} \right) W_{t_2, r_2} \left(\frac{n}{\hat{q}^2} \right) = \sum_{M, N \geq 1} F_{M, N}(m, n),$$

where the sums on M, N are over powers of 2 and

$$(3.12) \quad F_{M, N}(m, n) := f_{M, t_1, r_1}(m) f_{N, t_2, r_2}(n),$$

$$(3.13) \quad f_{X, t, r}(x) := \frac{1}{\sqrt{x}} W_{t, r} \left(\frac{x}{\hat{q}^2} \right) F_X(x).$$

The term (3.6) can be written as

$$(3.14) \quad T(c) = \sum_{M, N \geq 1} T_{M, N}(c),$$

$$(3.15) \quad T_{M, N}(c) = c \sum_{\substack{m, n \\ (q, mn)=1}} \tau_{1/2+ir_1}(m) \tau_{1/2+ir_2}(n) \\ \times S(m, n, c) F_{M, N}(m, n) J_{k-1} \left(\frac{4\pi\sqrt{mn}}{c} \right).$$

Lemma 3.6. *For any $\alpha \geq |t|$,*

$$(3.16) \quad x^i \frac{\partial^i}{\partial^i x} f_{X, t, r}(x) \ll_{\alpha, t, r} \frac{1}{\sqrt{X}} \left(\frac{x}{\hat{q}^2} \right)^{-\alpha} \quad \text{if } X \gg q^{1+\epsilon}$$

and

$$(3.17) \quad x^i \frac{\partial^i}{\partial^i x} f_{X, t, r}(x) \ll_{t, r} \frac{1}{\sqrt{X}} \left(\frac{x}{\hat{q}^2} \right)^{-|t|} \quad \text{if } X \ll q^{1+\epsilon}.$$

Proof. If $X \gg q^{1+\epsilon}$ we use (3.2) to get

$$x^i \frac{\partial^i}{\partial^i x} W_{t, r} \left(\frac{x}{\hat{q}^2} \right) \ll_{\alpha, t, r} \left(\frac{x}{\hat{q}^2} \right)^{-\alpha}.$$

If $X \ll q^{1+\epsilon}$ we use (3.3) to get

$$x^i \frac{\partial^i}{\partial^i x} W_{t, r} \left(\frac{x}{\hat{q}^2} \right) \ll_{t, r} \left(\frac{x}{\hat{q}^2} \right)^{-|t|}.$$

Finally, estimate (3.11) and Leibniz's rule yield the result. \square

Proposition 3.7. *For any $\epsilon > 0$, any $A > 0$ and $l = 0, 1$*

$$(3.18) \quad \sum_{\max(M, N) \gg q^{1+\epsilon}} \sum_{\substack{q/p^l | c}} \frac{1}{c^2} T_{M, N}(c) \ll_{\epsilon, A, r} q^{-A}.$$

Proof. Since $\max(M, N) \gg q^{1+\epsilon}$, there are three cases to consider:

- $M \gg q^{1+\epsilon}$, $N \ll q^{1+\epsilon}$;
- $M \ll q^{1+\epsilon}$, $N \gg q^{1+\epsilon}$;
- $M \gg q^{1+\epsilon}$, $N \gg q^{1+\epsilon}$.

We prove only the first case:

$$\begin{aligned} \sum_{\substack{M \gg q^{1+\epsilon} \\ N \ll q^{1+\epsilon}}} \sum_{\substack{q/p^l | c}} \frac{1}{c^2} T_{M, N}(c) &= \sum_{\substack{M \gg q^{1+\epsilon} \\ N \ll q^{1+\epsilon}}} \sum_{\substack{c, m, n \\ (q, mn)=1 \\ q/p^l | c}} \tau_{1/2+ir_1}(m) \tau_{1/2+ir_2}(n) \\ &\quad \times \frac{S(m, n, c)}{c} F_{M, N}(m, n) J_{k-1} \left(\frac{4\pi\sqrt{mn}}{c} \right). \end{aligned}$$

The sum over c can be decomposed into two cases

$$\begin{aligned} \sum_{q/p^l | c} \frac{S(m, n, c)}{c} J_{k-1} \left(\frac{4\pi\sqrt{mn}}{c} \right) &= \sum_{\substack{c < \sqrt{mn} \\ q/p^l | c}} \frac{S(m, n, c)}{c} J_{k-1} \left(\frac{4\pi\sqrt{mn}}{c} \right) \\ &\quad + \sum_{\substack{c \geq \sqrt{mn} \\ q/p^l | c}} \frac{S(m, n, c)}{c} J_{k-1} \left(\frac{4\pi\sqrt{mn}}{c} \right). \end{aligned}$$

By (A.4) and (2.20) for any $\delta > 0$ we have

$$\sum_{q/p^l | c} \frac{S(m, n, c)}{c} J_{k-1} \left(\frac{4\pi\sqrt{mn}}{c} \right) \ll (mn)^{3/4+\delta}.$$

We apply lemma 3.6 with $i = j = 0$:

$$\sum_{\substack{q/p^l | c}} \frac{1}{c^2} T_{M, N}(c) \ll_{\alpha_1, r} (MN)^{1/4+\delta} \left(\frac{q}{M} \right)^{\alpha_1} \left(\frac{q}{N} \right)^{|t_2|}.$$

Taking α_1 sufficiently large, we obtain that for any $\epsilon > 0$, any $A > 0$ and $l = 0, 1$

$$\sum_{\substack{M \gg q^{1+\epsilon} \\ N \ll q^{1+\epsilon}}} \sum_{\substack{q/p^l | c}} \frac{1}{c^2} T_{M, N}(c) \ll_{\epsilon, A, r} q^{-A}.$$

□

Corollary 3.8. *The range of summation in (3.14) can be restricted to $M, N \ll q^{1+\epsilon}$.*

Finally, we restrict the range of summation on c via large sieve inequality.

Lemma 3.9. *Let $l = 0, 1$. Assume that $M, N \ll q^{1+\epsilon}$. For any $C > \sqrt{MN}$ we have*

$$(3.19) \quad \sum_{\substack{c \geq C \\ \frac{q}{p^l} | c}} \frac{1}{c^2} T_{M,N}(c) \ll_{\epsilon, r} \left(\frac{\hat{q}^2}{M} \right)^{|t_1|} \left(\frac{\hat{q}^2}{N} \right)^{|t_2|} q^\epsilon \left(\frac{\sqrt{MN}}{C} \right)^{k-1-2\theta}.$$

Remark 3.10. Taking $C = \min \left(q^{\frac{1}{2-2\theta}} M^{1/2} N^{\frac{1-4\theta}{8-8\theta}}, q^{\frac{9-8\theta}{8-8\theta}} \right)$, the error term is bounded by $q^{\epsilon - \frac{k-1-2\theta}{8-8\theta}}$. See the proof of lemma 5.3 for the explanation of this choice.

Proof. We are going to apply theorem 2.3. In order to do so, we make a dyadic partition of the interval $[C, \infty)$ and assume that $c \in [C, 2C]$. By definition

$$\begin{aligned} \sum_{\substack{q/p^l | c}} \frac{1}{c^2} T_{M,N}(c) &= \sum_{\substack{n, m \\ (q, nm)=1}} \sum_{\substack{q/p^l | c}} \tau_{1/2+ir_1}(m) \tau_{1/2+ir_2}(n) \frac{1}{c} S(m, n, c) \\ &\times J_{k-1} \left(\frac{4\pi\sqrt{mn}}{c} \right) F_{M,N}(m, n) = \frac{p^l}{q} \sum_{\substack{n, m \\ (q, nm)=1}} \tau_{1/2+ir_1}(m) \tau_{1/2+ir_2}(n) \\ &\times \sum_{c_1} \frac{1}{c_1} S(m, n, c_1 q/p^l) J_{k-1} \left(\frac{4\pi\sqrt{mn} p^l}{c_1 q} \right) F_{M,N}(m, n). \end{aligned}$$

Here $m \in [M/2, 3M]$, $n \in [N/2, 3N]$ and $c_1 \in [C_1, 2C_1]$ with $C_1 := Cp^l/q$. Let

$$X := \left(\frac{\hat{q}^2}{M} \right)^{-|t_1|} \left(\frac{\hat{q}^2}{N} \right)^{-|t_2|} \sqrt{MN} C_1 \left(\frac{\sqrt{MN}}{C} \right)^{-k+1}.$$

As a test function we choose

$$g(m, n, c_1) := \frac{X}{c_1} F_{M,N}(m, n) J_{k-1} \left(\frac{4\pi\sqrt{mn} p^l}{c_1 q} \right).$$

It satisfies condition (2.21), and theorem 2.3 can be applied with $r = 1$ and $s = q/p^l$. Hence

$$\sum_{\substack{\frac{q}{p^l} | c \\ c \geq C}} \frac{1}{c^2} T_{M,N}(c) \ll_{\epsilon, r} \left(\frac{\hat{q}^2}{M} \right)^{|t_1|} \left(\frac{\hat{q}^2}{N} \right)^{|t_2|} q^\epsilon \left(\frac{\sqrt{MN}}{C} \right)^{k-1-2\theta}.$$

□

3.4 Removing the coprimality condition

In order to apply theorem 2.8, we have to exclude the coprimality condition in $T_{M,N}(c)$. This can be done using the criterion of vanishing of classical Kloosterman sum given by lemma 2.2. Let

$$(3.20) \quad f(m, n, c) := F_{M,N}(m, n) J_{k-1} \left(\frac{4\pi\sqrt{mn}}{c} \right).$$

Proposition 3.11. *Let m, n, c be three strictly positive integers and p be a prime number. Suppose $p^2 | c$. Then*

$$\begin{aligned} \sum_{(q, mn)=1} \tau_{1/2+ir_1}(m) \tau_{1/2+ir_2}(n) S(m, n, c) f(m, n, c) = \\ \sum_{m, n \geq 1} \tau_{1/2+ir_1}(m) \tau_{1/2+ir_2}(n) S(m, n, c) f(m, n, c) \\ - \tau_{1/2+ir_2}(p) \sum_{m, n \geq 1} \tau_{1/2+ir_1}(m) \tau_{1/2+ir_2}(n) S(m, np, c) f(m, np, c) \\ + \sum_{m, n \geq 1} \tau_{1/2+ir_1}(m) \tau_{1/2+ir_2}(n) S(m, np^2, c) f(m, np^2, c). \end{aligned}$$

Proof. Recall that $q = p^\nu$. Therefore,

$$\sum_{(q, mn)=1} = \sum_{p \nmid mn} = \sum_{m, n} - \sum_{p | mn} = \sum_{m, n} - \sum_{p | n} - \sum_{p | m, p \nmid n}.$$

The sum

$$\sum_{p | m, p \nmid n} \tau_{1/2+ir_1}(m) \tau_{1/2+ir_2}(n) S(m, n, c) f(m, n, c) = 0$$

since the Kloosterman sum vanishes by lemma 2.2. Further,

$$\begin{aligned} \sum_{p | n} \tau_{1/2+ir_1}(m) \tau_{1/2+ir_2}(n) S(m, n, c) f(m, n, c) \\ = \sum_n \tau_{1/2+ir_1}(m) \tau_{1/2+ir_2}(np) S(m, np, c) f(m, np, c). \end{aligned}$$

The identity (2.27) implies that

$$\tau_{1/2+ir_2}(np) = \tau_{1/2+ir_2}(p) \tau_{1/2+ir_2}(n) - \tau_{1/2+ir_2} \left(\frac{n}{p} \right) \text{ if } (p, n) = p,$$

$$\tau_{1/2+ir_2}(np) = \tau_{1/2+ir_2}(p) \tau_{1/2+ir_2}(n) \text{ if } (p, n) = 1.$$

This yields the result. \square

3.5 Applying the Poisson-type summation formula

By proposition 3.11, the term (3.15) can be decomposed as follows

$$(3.21) \quad T_{M,N}(c) = TS(c, 0) - \tau_{1/2+ir_2}(p)TS(c, 1) + TS(c, 2),$$

where

$$(3.22) \quad TS(c, B) = c \sum_{m,n \geq 1} \tau_{1/2+ir_1}(m) \tau_{1/2+ir_2}(n) S(m, np^B, c) f(m, np^B, c)$$

with $B = 0, 1, 2$ and $f(m, n, c)$ is defined by (3.20).

Proposition 3.12. *One has*

$$(3.23) \quad TS(c, B) = TS^*(c, B) + TS^+(c, B) + TS^-(c, B),$$

where

$$(3.24) \quad TS^*(c, B) = \sum_{n \geq 1} \tau_{1/2+ir_2}(n) S(0, np^B, c) [G_{r_1}^*(np^B) + G_{-r_1}^*(np^B)],$$

$$(3.25) \quad TS^\mp(c, B) = \sum_{m,n \geq 1} \tau_{1/2+ir_1}(m) \tau_{1/2+ir_2}(n) \times S(0, np^B \mp m, c) G_{r_1}^\mp(m, np^B).$$

The functions G_r^* , G_r^- , G_r^+ are defined as follows

$$(3.26) \quad G_r^*(y) = \frac{\zeta(1+2ir)}{c^{2ir}} \int_0^\infty J_{k-1} \left(\frac{4\pi\sqrt{xy}}{c} \right) F_{M,N}(x, y) x^{ir} dx,$$

$$(3.27) \quad G_r^-(z, y) = 2\pi \int_0^\infty k_0 \left(\frac{4\pi\sqrt{xz}}{c}, 1/2 + ir \right) \times J_{k-1} \left(\frac{4\pi\sqrt{xy}}{c} \right) F_{M,N}(x, y) dx,$$

$$(3.28) \quad G_r^+(z, y) = 2\pi \int_0^\infty k_1 \left(\frac{4\pi\sqrt{xz}}{c}, 1/2 + ir \right) \times J_{k-1} \left(\frac{4\pi\sqrt{xy}}{c} \right) F_{M,N}(x, y) dx.$$

Proof. The function f is smooth, compactly supported, and thus satisfies all conditions of theorem 2.8. Applying the summation formula with $\phi(x) := f(\frac{c^2}{16\pi^2}x^2, np^B, c)$, we obtain

$$\begin{aligned} & \sum_{m \geq 1} e\left(\frac{md}{c}\right) \tau_{1/2+ir_1}(m) f(m, np^B, c) = \\ & \frac{\zeta(1+2ir_1)}{c^{1+2ir_1}} \int_0^\infty f(x, np^B, c) x^{ir_1} dx + \frac{\zeta(1-2ir_1)}{c^{1-2ir_1}} \int_0^\infty f(x, np^B, c) x^{-ir_1} dx \\ & + \frac{2\pi}{c} \sum_{m \geq 1} \tau_{1/2+ir_1}(m) \int_0^\infty e\left(\frac{-m\bar{d}}{c}\right) k_0\left(\frac{4\pi}{c}\sqrt{xm}, 1/2+ir_1\right) f(x, np^B, c) dx \\ & + \frac{2\pi}{c} \sum_{m \geq 1} \tau_{1/2+ir_1}(m) \int_0^\infty e\left(\frac{m\bar{d}}{c}\right) k_1\left(\frac{4\pi}{c}\sqrt{xm}, 1/2+ir_1\right) f(x, np^B, c) dx. \end{aligned}$$

Plugging this in (3.22) yields the assertion. \square

The next lemma shows that $TS^*(c)$ term contributes to the fourth moment as an error.

Lemma 3.13. *Let $l = 0, 1$. Then*

$$(3.29) \quad \sum_{\substack{\frac{q}{p^l}|c \\ M, N \leq q^{1+\epsilon}}} c^{-2} TS^*(c, B) \ll_{\epsilon, \mathbf{r}} q^{-1+\epsilon}.$$

Proof. We use lemma 3.6 to estimate $F_{M,N}(m, n)$. The J -Bessel function can be trivially bounded by 1. Then

$$G_r^*(np^B) \ll_{\mathbf{r}} \left(\frac{\hat{q}^2}{M}\right)^{|t_1|} \left(\frac{\hat{q}^2}{N}\right)^{|t_2|} \left(\frac{M}{N}\right)^{1/2}.$$

Since $S(0, np^B, c) \ll (np^B, c)$, we have

$$TS^*(c, B) \ll_{\mathbf{r}} (MN)^{1/2} q^\epsilon \left(\frac{\hat{q}^2}{M}\right)^{|t_1|} \left(\frac{\hat{q}^2}{N}\right)^{|t_2|}.$$

Therefore,

$$\sum_{\substack{\frac{q}{p^l}|c \\ M, N \leq q^{1+\epsilon}}} c^{-2} TS^*(c, B) \ll_{\epsilon, \mathbf{r}} q^{-1+\epsilon}.$$

\square

The last two summands require more detailed treatment. We rewrite the sums TS^\pm in the form that is more convenient for later computations

$$\begin{aligned} TS^-(c, B) &= \sum_{m \geq 1} \sum_{n \geq 1} \tau_{1/2+ir_1}(m) \tau_{1/2+ir_2}(n) S(0, np^B - m, c) G_{r_1}^-(m, np^B) \\ &= \phi(c) \sum_{n \geq 1} \tau_{1/2+ir_1}(np^B) \tau_{1/2+ir_2}(n) G_{r_1}^-(np^B, np^B) + \sum_{h \neq 0} S(0, h, c) T_h^-(c, B) \end{aligned}$$

and

$$\begin{aligned} TS^+(c, B) &= \sum_{m \geq 1} \sum_{n \geq 1} \tau_{1/2+ir_1}(m) \tau_{1/2+ir_2}(n) S(0, np^B + m, c) G_{r_1}^+(m, np^B) \\ &= \sum_{h \neq 0} S(0, h, c) T_h^+(c, B), \end{aligned}$$

where

$$(3.30) \quad T_h^\mp(c, B) = \sum_{m \mp np^B = h} \tau_{1/2+ir_1}(m) \tau_{1/2+ir_2}(n) G_{r_1}^\mp(m, p^B n).$$

At this point, the non-diagonal term splits into off-diagonal (corresponds to $h = 0$) and off-off-diagonal ($h \neq 0$) parts.

Theorem 3.14. *One has*

$$(3.31) \quad M^{OD} = M^{OD}(0) - \tau_{1/2+ir_2}(p) M^{OD}(1) + M^{OD}(2),$$

$$(3.32) \quad M^{OOD} = M^{OOD}(0) - \tau_{1/2+ir_2}(p) M^{OOD}(1) + M^{OOD}(2).$$

For $B = 0, 1, 2$

$$\begin{aligned} M^{OD}(B) &= 2\pi i^{-k} \left(\sum_{\substack{q|c \\ c \ll C}} \frac{\phi(c)}{c^2} \sum_{\substack{n \\ M, N \ll q^{1+\epsilon}}} \tau_{1/2+ir_1}(np^B) \tau_{1/2+ir_2}(n) G_{r_1}^-(np^B, np^B) \right. \\ &\quad \left. - \frac{1}{p} \sum_{\substack{\frac{q}{p}|c \\ c \ll C}} \frac{\phi(c)}{c^2} \sum_{\substack{n \\ M, N \ll q^{1+\epsilon}}} \tau_{1/2+ir_1}(np^B) \tau_{1/2+ir_2}(n) G_{r_1}^-(np^B, np^B) \right), \end{aligned}$$

$$\begin{aligned} M^{OOD}(B) &= 2\pi i^{-k} \left(\sum_{\substack{q|c \\ c \ll C}} \frac{1}{c^2} \sum_{M, N \ll q^{1+\epsilon}} \sum_{h \neq 0} S(0, h, c) (T_h^-(c, B) + T_h^+(c, B)) \right. \\ &\quad \left. - \frac{1}{p} \sum_{\substack{\frac{q}{p}|c \\ c \ll C}} \frac{1}{c^2} \sum_{M, N \ll q^{1+\epsilon}} \sum_{h \neq 0} S(0, h, c) (T_h^-(c, B) + T_h^+(c, B)) \right). \end{aligned}$$

Here $T_h^\pm(c, B)$ is given by (3.30) and $G_r^\pm(z, y)$ by (3.27), (3.28).

4 Asymptotic evaluation of the diagonal and off-diagonal terms

The main result of this section is the asymptotic formula for the diagonal and off-diagonal terms.

Theorem 4.1. *Up to a negligible error term, we have*

$$(4.1) \quad M^D + M^{OD} = \frac{\phi(q)}{q} \sum_{\epsilon_1, \epsilon_2 = \pm 1} \hat{q}^{-2t_1 - 2t_2 + 2\epsilon_1 t_1 + 2\epsilon_2 t_2} \frac{\Gamma(\epsilon_1 t_1 + ir_1 + k/2)}{\Gamma(t_1 + ir_1 + k/2)} \\ \times \frac{\Gamma(\epsilon_1 t_1 - ir_1 + k/2)}{\Gamma(t_1 - ir_1 + k/2)} \frac{\Gamma(\epsilon_2 t_2 + ir_2 + k/2) \Gamma(\epsilon_2 t_2 - ir_2 + k/2)}{\Gamma(t_2 + ir_2 + k/2) \Gamma(t_2 - ir_2 + k/2)} \\ \times \zeta_q(1 + 2\epsilon_1 t_1) \zeta_q(1 + 2\epsilon_2 t_2) \frac{\prod_{\epsilon_3, \epsilon_4 = \pm 1} \zeta_q(1 + \epsilon_1 t_1 + \epsilon_2 t_2 + i\epsilon_3 r_1 + i\epsilon_4 r_2)}{\zeta_q(2 + 2\epsilon_1 t_1 + 2\epsilon_2 t_2)}.$$

4.1 Extension of summations

First, we reintroduce the summation over $c > C$ and $\max(M, N) \gg q^{1+\epsilon}$ for the off-diagonal term at the cost of admissible error.

Proposition 4.2. *For any $\epsilon > 0$*

$$(4.2) \quad \sum_{\substack{\frac{q}{p^t}|c \\ c > C}} \frac{\phi(c)}{c^2} \sum_{\max(M, N) \ll q^{1+\epsilon}} \sum_n \tau_{1/2+ir_1}(np^B) \tau_{1/2+ir_2}(n) G_{r_1}^-(np^B, np^B) \\ \ll_{\epsilon, r} q^{\epsilon - \frac{k-1}{8-8\theta}}.$$

Proof. Let

$$\eta_C(c) = \begin{cases} 0 & \text{if } c > C \\ 1 & \text{if } c \leq C. \end{cases}$$

Consider

$$T_1 := \sum_{\substack{\frac{q}{p^t}|c \\ c > C}} \frac{\phi(c)}{c^2} \sum_{\max(M, N) \ll q^{1+\epsilon}} \sum_n \tau_{1/2+ir_1}(np^B) \tau_{1/2+ir_2}(n) G_{r_1}^-(np^B, np^B) \\ = \sum_{\max(M, N) \ll q^{1+\epsilon}} \sum_n \tau_{1/2+ir_1}(np^B) \tau_{1/2+ir_2}(n) \int_0^\infty k_0\left(4\pi\sqrt{xn p^B}, 1/2 + ir_1\right) \\ \times 2\pi J_{k-1}\left(4\pi\sqrt{xn p^B}\right) \sum_{\frac{q}{p^t}|c} (1 - \eta_C(c)) \phi(c) F_{M, N}(xc^2, np^B) dx.$$

We use lemma 3.6 to bound $F_{M,N}(xc^2, np^B)$, formula (A.4) to bound the Bessel function $J_{k-1}\left(4\pi\sqrt{xn}p^B\right)$ and trivial estimate for the Bessel kernel

$$k_0\left(4\pi\sqrt{xn}p^B, 1/2 + ir_1\right) \ll 1.$$

Then

$$T_1 \ll_{\epsilon, \mathbf{r}} q^\epsilon \sum_{M, N \ll q^{1+\epsilon}} \frac{1}{\sqrt{MN}} \sum_{n \sim N} \int_0^{2M/C^2} (\sqrt{xn})^{k-1} \frac{M}{qx} dx \ll_{\epsilon, \mathbf{r}} q^{\epsilon - \frac{k-1}{8-8\theta}}.$$

□

Proposition 4.3. *For any $\epsilon > 0$, any $A > 0$ and $l = 0, 1$*

$$(4.3) \quad \sum_{\frac{q}{p^l} | c} \frac{\phi(c)}{c^2} \sum_{\max(M, N) \gg q^{1+\epsilon}} \sum_n \tau_{1/2+ir_1}(np^B) \tau_{1/2+ir_2}(n) G_{r_1}^-(np^B, np^B) \ll_{\epsilon, A, \mathbf{r}} q^{-A}.$$

Proof. The statement can be proved analogously to proposition 3.7. □

Now it is possible to combine all functions F_M into F and replace $\sum_{M, N} F_{M, N}$ by

$$(4.4) \quad F(x, y) := \frac{1}{\sqrt{xy}} W_{t_1, r_1} \left(\frac{x}{\hat{q}^2} \right) W_{t_2, r_2} \left(\frac{y}{\hat{q}^2} \right) F(x) F(y),$$

where $F(x)$ is a smooth function, compactly supported in $[1/2, \infty)$ such that $F(x) = 1$ for $x \geq 1$.

Proposition 4.4. *Up to the error term $O_{\mathbf{r}, \epsilon}(q^{\epsilon-k/2})$, the product $F(x)F(y)$ can be replaced by 1 in (4.4).*

Proof. Consider

$$\begin{aligned} T_2 &:= \sum_{\frac{c}{q} | c} \frac{\phi(c)}{c^2} \sum_n \tau_{1/2+ir_1}(np^B) \tau_{1/2+ir_2}(n) \int_0^1 k_0 \left(\frac{4\pi\sqrt{xn}p^B}{c}, 1/2 + ir_1 \right) \\ &\times J_{k-1} \left(\frac{4\pi\sqrt{xn}p^B}{c} \right) \frac{1}{\sqrt{xn}p^B} W_{t_1, r_1} \left(\frac{x}{\hat{q}^2} \right) W_{t_2, r_2} \left(\frac{np^B}{\hat{q}^2} \right) (1 - F(x)) dx. \end{aligned}$$

We estimate the kernel $k_0\left(\frac{4\pi\sqrt{xn}p^B}{c}, 1/2 + ir_1\right)$ trivially by 1 and apply the following bound for the J -Bessel function

$$J_{k-1} \left(\frac{4\pi\sqrt{xn}p^B}{c} \right) \ll \left(\frac{\sqrt{xn}}{c} \right)^{k-1}.$$

If $n < q$, the function W_{t_2, r_2} can be estimated using (3.3). Otherwise we apply (3.2). This gives

$$T_2 \ll_r q^{\epsilon - k/2}.$$

□

4.2 Asymptotics of diagonal and off-diagonal terms

The off-diagonal term can be written as

$$M^{OD}(B) = \hat{q}^{-2t_1 - 2t_2} \sum_n \frac{\tau_{1/2+ir_1}(np^B) \tau_{1/2+ir_2}(n)}{np^B} W_{t_2, r_2} \left(\frac{np^B}{\hat{q}^2} \right) Z(np^B)$$

for $B = 0, 1, 2$ with

$$\begin{aligned} Z(u) &:= 2\pi i^{-k} \int_0^\infty k_0(z, 1/2 + ir_1) J_{k-1}(z) \\ &\times \left(\sum_{q|c} \frac{\phi(c)}{c} W_{t_1, r_1} \left(\frac{z^2 c^2}{(4\pi)^2 \hat{q}^2 u} \right) - \frac{1}{p} \sum_{\frac{q}{p}|c} \frac{\phi(c)}{c} W_{t_1, r_1} \left(\frac{z^2 c^2}{(4\pi)^2 \hat{q}^2 u} \right) \right) dz. \end{aligned}$$

Note that we made the change of variables $x = \frac{z^2 c^2}{(4\pi)^2 u}$ in the integral. Applying (3.1), we have

$$\begin{aligned} Z(u) &= 2\pi i^{-k} \int_0^\infty k_0(z, 1/2 + ir_1) J_{k-1}(z) \\ &\times \frac{1}{2\pi i} \int_{(3)} P_r(s) \frac{\Gamma(s + ir_1 + k/2) \Gamma(s - ir_1 + k/2)}{\Gamma(t_1 + ir_1 + k/2) \Gamma(t_1 - ir_1 + k/2)} \\ &\times \zeta_q(1 + 2s) \left(\frac{z^2}{(4\pi)^2 \hat{q}^2 u} \right)^{-s} \times \left[\sum_{q|c} \frac{\phi(c)}{c^{1+2s}} - \frac{1}{p} \sum_{\frac{q}{p}|c} \frac{\phi(c)}{c^{1+2s}} \right] \frac{2s ds}{s^2 - t_1^2} dz. \end{aligned}$$

The term in the brackets can be simplified

$$\sum_{q|c} \frac{\phi(c)}{c^{1+2s}} - \frac{1}{p} \sum_{\frac{q}{p}|c} \frac{\phi(c)}{c^{1+2s}} = \frac{\phi(q)}{q^{1+2s}} \frac{1 - p^{2s-1}}{1 - p^{-2s}} \frac{\zeta_q(2s)}{\zeta_q(2s+1)}.$$

Lemma A.3 implies that

$$\begin{aligned} \int_0^\infty k_0(z, 1/2 + ir_1) J_{k-1}(z) z^{-2s} dz &= \frac{\Gamma(2s)}{2^{2s+1} \cos(\pi(1/2 + ir_1))} \\ &\times \left(\frac{\Gamma(ir_1 + k/2 - s)}{\Gamma(-ir_1 + k/2 + s) \Gamma(ir_1 + k/2 + s) \Gamma(ir_1 - k/2 + s + 1)} \right. \\ &\quad \left. - \frac{\Gamma(-ir_1 + k/2 - s)}{\Gamma(-ir_1 + k/2 + s) \Gamma(ir_1 + k/2 + s) \Gamma(-ir_1 - k/2 + s + 1)} \right). \end{aligned}$$

By duplication and reflection formulas

$$\begin{aligned} \int_0^\infty k_0(z, 1/2 + ir_1) J_{k-1}(z) z^{-2s} dz &= -\frac{i^k \Gamma(s) \Gamma(s + 1/2)}{2^2 \pi^{3/2} \sin(\pi ir_1)} \\ &\times \frac{\Gamma(ir_1 + k/2 - s) \Gamma(-ir_1 + k/2 - s)}{\Gamma(ir_1 + k/2 + s) \Gamma(-ir_1 + k/2 + s)} [\sin \pi(-s - ir_1) - \sin \pi(-s + ir_1)]. \end{aligned}$$

Observe that

$$\frac{\Gamma(1/2 - s) \Gamma(1/2 + s)}{2\pi \sin(\pi ir_1)} [\sin(\pi(-s - ir_1)) - \sin(\pi(-s + ir_1))] = -1.$$

Consequently,

$$\begin{aligned} Z(u) &= \frac{\phi(q)}{q} \frac{1}{2\pi i} \int_{(3)} P_r(s) \frac{\Gamma(-s + ir_1 + k/2) \Gamma(-s - ir_1 + k/2)}{\Gamma(t_1 + ir_1 + k/2) \Gamma(t_1 - ir_1 + k/2)} \\ &\quad \times \zeta_q(1 - 2s) \left(\frac{u}{\hat{q}^2} \right)^s \frac{2s ds}{s^2 - t_1^2}. \end{aligned}$$

Shifting the contour of integration to $\Re(s) = -3$, we cross poles at $s = \pm t_1$.

Hence

$$\begin{aligned} Z(u) &= \frac{\phi(q)}{q} \sum_{\epsilon_1 = \pm 1} \frac{\Gamma(\epsilon_1 t_1 + ir_1 + k/2) \Gamma(\epsilon_1 t_1 - ir_1 + k/2)}{\Gamma(t_1 + ir_1 + k/2) \Gamma(t_1 - ir_1 + k/2)} \\ &\quad \times \zeta_q(1 + 2\epsilon_1 t_1) \left(\frac{u}{\hat{q}^2} \right)^{-\epsilon_1 t_1} - \frac{\phi(q)}{q} \frac{1}{2\pi i} \int_{(3)} P_r(s) \\ &\quad \times \frac{\Gamma(s + ir_1 + k/2) \Gamma(s - ir_1 + k/2)}{\Gamma(t_1 + ir_1 + k/2) \Gamma(t_1 - ir_1 + k/2)} \zeta_q(1 + 2s) \left(\frac{u}{\hat{q}^2} \right)^{-s} \frac{2s ds}{s^2 - t_1^2}. \end{aligned}$$

Substitution of $Z(np^B)$ into $M^{OD}(B)$ gives

$$\begin{aligned} M^{OD}(B) &= \frac{\phi(q)}{q} \hat{q}^{-2t_1 - 2t_2} \sum_n \frac{\tau_{1/2+ir_1}(np^B) \tau_{1/2+ir_2}(n)}{np^B} W_{t_2, r_2} \left(\frac{np^B}{\hat{q}^2} \right) \\ &\quad \times \left(-W_{t_1, r_1} \left(\frac{np^B}{\hat{q}^2} \right) + \sum_{\epsilon_1 = \pm 1} \frac{\Gamma(\epsilon_1 t_1 + ir_1 + k/2) \Gamma(\epsilon_1 t_1 - ir_1 + k/2)}{\Gamma(t_1 + ir_1 + k/2) \Gamma(t_1 - ir_1 + k/2)} \right. \\ &\quad \left. \times \zeta_q(1 + 2\epsilon_1 t_1) \left(\frac{np^B}{\hat{q}^2} \right)^{-\epsilon_1 t_1} \right). \end{aligned}$$

Property of multiplicity (2.27) implies that

$$\begin{aligned} \sum_{\substack{n \geq 1 \\ (n, p) = 1}} \tau_{1/2+ir_2}(n) f(n) &= \sum_{n \geq 1} \tau_{1/2+ir_2}(n) f(n) \\ &\quad - \tau_{1/2+ir_2}(p) \sum_{n \geq 1} \tau_{1/2+ir_2}(n) f(np) + \sum_{n \geq 1} \tau_{1/2+ir_2}(n) f(np^2). \end{aligned}$$

Thus,

$$(4.5) \quad M^D + M^{OD} = \frac{\phi(q)}{q} \hat{q}^{-2t_1-2t_2} \sum_{(n,p)=1} \frac{\tau_{1/2+ir_1}(n) \tau_{1/2+ir_2}(n)}{n} W_{t_2, r_2} \left(\frac{n}{\hat{q}^2} \right) \\ \times \left(\sum_{\epsilon_1=\pm 1} \frac{\Gamma(\epsilon_1 t_1 + ir_1 + k/2) \Gamma(\epsilon_1 t_1 - ir_1 + k/2)}{\Gamma(t_1 + ir_1 + k/2) \Gamma(t_1 - ir_1 + k/2)} \zeta_q(1 + 2\epsilon_1 t_1) \left(\frac{n}{\hat{q}^2} \right)^{-\epsilon_1 t_1} \right).$$

Ramanujan's identity (2.28) yields

$$\sum_{(n,p)=1} \frac{\tau_{1/2+ir_1}(n) \tau_{1/2+ir_2}(n)}{n^{1+\epsilon_1 t_1+s}} = \frac{\prod_{\epsilon_3, \epsilon_4=\pm 1} \zeta_q(1 + \epsilon_1 t_1 + s + i\epsilon_3 r_1 + i\epsilon_4 r_2)}{\zeta_q(2 + 2\epsilon_1 t_1 + 2s)}.$$

Therefore,

$$M^D + M^{OD} = \frac{\phi(q)}{q} \sum_{\epsilon_1=\pm 1} \frac{\Gamma(\epsilon_1 t_1 + ir_1 + k/2) \Gamma(\epsilon_1 t_1 - ir_1 + k/2)}{\Gamma(t_1 + ir_1 + k/2) \Gamma(t_1 - ir_1 + k/2)} \\ \times \hat{q}^{-2t_1-2t_2+2\epsilon_1 t_1} \frac{1}{2\pi i} \int_{\Re s=3} \frac{P_r(s)}{P_r(t_2)} \hat{q}^{2s} \zeta_q(1 + 2s) \zeta_q(1 + 2\epsilon_1 t_1) \\ \times \frac{\Gamma(s + ir_2 + k/2) \Gamma(s - ir_2 + k/2)}{\Gamma(t_2 + ir_2 + k/2) \Gamma(t_2 - ir_2 + k/2)} \\ \times \frac{\prod_{\epsilon_3, \epsilon_4=\pm 1} \zeta_q(1 + \epsilon_1 t_1 + s + i\epsilon_3 r_1 + i\epsilon_4 r_2)}{\zeta_q(2 + 2\epsilon_1 t_1 + 2s)} \frac{2s ds}{s^2 - t_2^2}.$$

Shifting the contour of integration to $\Re s = -1/2$, the resulting integral is bounded by $q^{\epsilon-1/2}$ plus the contribution of simple poles at $s = \pm t_2$. Up to an error term,

$$M^D + M^{OD} = \frac{\phi(q)}{q} \sum_{\epsilon_1, \epsilon_2=\pm 1} \hat{q}^{-2t_1-2t_2+2\epsilon_1 t_1+2\epsilon_2 t_2} \frac{\Gamma(\epsilon_1 t_1 + ir_1 + k/2)}{\Gamma(t_1 + ir_1 + k/2)} \\ \times \frac{\Gamma(\epsilon_1 t_1 - ir_1 + k/2)}{\Gamma(t_1 - ir_1 + k/2)} \frac{\Gamma(\epsilon_2 t_2 + ir_2 + k/2) \Gamma(\epsilon_2 t_2 - ir_2 + k/2)}{\Gamma(t_2 + ir_2 + k/2) \Gamma(t_2 - ir_2 + k/2)} \\ \times \zeta_q(1 + 2\epsilon_1 t_1) \zeta_q(1 + 2\epsilon_2 t_2) \frac{\prod_{\epsilon_3, \epsilon_4=\pm 1} \zeta_q(1 + \epsilon_1 t_1 + \epsilon_2 t_2 + i\epsilon_3 r_1 + i\epsilon_4 r_2)}{\zeta_q(2 + 2\epsilon_1 t_1 + 2\epsilon_2 t_2)}.$$

By letting shifts tend to zero in (4.5), we find

$$(4.6) \quad M^D + M^{OD} = \left(\frac{\phi(q)}{q} \right)^2 \sum_{(n,p)=1} \frac{\tau(n)^2}{n} W_{0,0} \left(\frac{n}{\hat{q}^2} \right) \log \left(\frac{\hat{q}^2}{n} \right).$$

The equality (2.29) gives

$$M^D + M^{OD} = \frac{1}{2\pi i} \left(\frac{\phi(q)}{q} \right)^2 \int_{(3)} \frac{P_r(s)}{P_r(0)} \frac{\Gamma(k/2 + s)^2}{\Gamma(k/2)^2} \zeta_q(1 + 2s) \hat{q}^{2s} \frac{\zeta_q(1 + s)^4}{\zeta_q(2 + 2s)} \\ \times \left[\log \hat{q}^2 + 4 \frac{\zeta'_q}{\zeta_q}(1 + s) - 2 \frac{\zeta'_q}{\zeta_q}(2 + 2s) \right] \frac{2ds}{s}.$$

Shifting the contour of integration to $\Re s = -1/2$, the resulting integral is bounded by $q^{-1/2}$ plus the contribution of multiple poles at $s = 0$. Calculation of the residue

$$\left(\frac{\phi(q)}{q}\right)^7 \frac{1}{\zeta_q(2)} \operatorname{Res}_{s=0} \frac{\hat{q}^{2s}}{s^6} \left(\log \hat{q} - \frac{4}{s}\right)$$

shows that the main term is

$$\left(\frac{\phi(q)}{q}\right)^7 \frac{p^2}{p^2 - 1} \frac{(\log q)^6}{60\pi^2}.$$

5 Off-off-diagonal term: double integral representation

In this section, we will show that the off-off-diagonal main term can be written as a double integral.

Theorem 5.1. *Up to a negligible error, we have*

$$(5.1) \quad M^{OOD} = \frac{\phi(q)}{q} \sum_{\epsilon_1, \epsilon_2 = \pm 1} \frac{\zeta_q(1 + 2i\epsilon_1 r_1) \zeta_q(1 + 2i\epsilon_2 r_2)}{\zeta_q(2 + 2i\epsilon_1 r_1 + 2i\epsilon_2 r_2)} \hat{q}^{-2t_1 - 2t_2} \\ \times \hat{q}^{-2i\epsilon_1 r_1 + 2i\epsilon_2 r_2} \frac{1}{(2\pi i)^2} \int_{\Re t = k/2 + 0.7} \int_{\Re s = k/2 - 0.4} I_{\epsilon_1, \epsilon_2}(s, t) \frac{2s ds}{s^2 - t_1^2} \frac{2t dt}{t^2 - t_2^2},$$

where

$$(5.2) \quad I_{\epsilon_1, \epsilon_2}(s, t) = \frac{P_r(s) P_r(t)}{P_r(t_1) P_r(t_2)} \zeta_q(1 + t + s + i\epsilon_1 r_1 + i\epsilon_2 r_2) \\ \times \zeta_q(1 + t - s + i\epsilon_1 r_1 + i\epsilon_2 r_2) \zeta_q(1 - t + s + i\epsilon_1 r_1 + i\epsilon_2 r_2) \zeta_q(1 - t - s + i\epsilon_1 r_1 + i\epsilon_2 r_2) \\ \times \frac{\Gamma(k/2 + s + i\epsilon_1 r_1) \Gamma(k/2 + t + i\epsilon_2 r_2)}{\Gamma(k/2 + t_1 + ir_1) \Gamma(k/2 + t_1 - ir_1)} \frac{\Gamma(k/2 - s + i\epsilon_1 r_1) \Gamma(k/2 - t + i\epsilon_2 r_2)}{\Gamma(k/2 + t_2 + ir_2) \Gamma(k/2 + t_2 - ir_2)}.$$

5.1 Estimation of $G_{r_1}^\mp$

The expression

$$T_h^\pm(c, B) = \sum_{m \pm np^B = h} \tau_{1/2 + ir_1}(m) \tau_{1/2 + ir_2}(n) G_{r_1}^\pm(m, p^B n)$$

can be evaluated using theorem 2.9. To this end, we show that the functions $G_{r_1}^\pm$, defined by (3.27) and (3.28), satisfy condition (2.33).

Let $Q := 1 + \frac{\sqrt{MN}}{c}$, $Z := \frac{Q^2 c^2}{M}$, $Y := N$.

Lemma 5.2. *For all positive n_1 and n_2*

$$(5.3) \quad z^j y^i \frac{\partial^j}{\partial z^j} \frac{\partial^i}{\partial y^i} G_{r_1}^\pm(z, y) \ll (1 + \frac{z}{Z})^{-n_1} (1 + \frac{y}{Y})^{-n_2} \frac{M^{1/2}}{N^{1/2}} \\ \times \left(\frac{\sqrt{MN}}{c} \right)^{k-1} Q^{j+i-k+1/2}.$$

Proof. Consider

$$G_{r_1}^-(z, y) = 2\pi \int_0^\infty k_0 \left(\frac{4\pi\sqrt{xz}}{c}, 1/2 + ir_1 \right) J_{k-1} \left(\frac{4\pi\sqrt{xy}}{c} \right) \\ \times F_{M,N}(x, y) dx = -\frac{\pi}{\sin \pi ir_1} \int_0^\infty \left[J_{2ir_1} \left(\frac{4\pi\sqrt{xz}}{c} \right) - J_{-2ir_1} \left(\frac{4\pi\sqrt{xz}}{c} \right) \right] \\ \times J_{k-1} \left(\frac{4\pi\sqrt{xy}}{c} \right) F_{M,N}(x, y) dx.$$

Suppose that $z > Z$. Let $u := \frac{4\pi\sqrt{xz}}{c}$. Then

$$G_{r_1}^-(z, y) = -\frac{c^2}{8\pi z \sin \pi ir_1} \int_0^\infty u [J_{2ir_1}(u) - J_{-2ir_1}(u)] \\ \times J_{k-1} \left(u \sqrt{\frac{y}{z}} \right) F_{M,N} \left(\frac{c^2 u^2}{16\pi^2 z}, y \right) du.$$

It is sufficient to estimate

$$G_1(z, y) := -\frac{c^2}{8\pi z \sin \pi ir_1} \int_0^\infty u J_{2ir_1}(u) J_{k-1} \left(u \sqrt{\frac{y}{z}} \right) \\ \times F_{M,N} \left(\frac{c^2 u^2}{16\pi^2 z}, y \right) du.$$

Note that $F_{M,N}(x, y)$ is compactly supported on $[M/2, 3M] \times [N/2, 3N]$. Let

$$f(u) := g_1(u) g_2(u) u^{-2ir_1}$$

with

$$g_1(u) := J_{k-1} \left(u \sqrt{\frac{y}{z}} \right) \text{ and } g_2(u) := F_{M,N} \left(\frac{c^2 u^2}{16\pi^2 z}, y \right).$$

The recurrent relation (A.1) implies that

$$G_1(z, y) = -\frac{c^2}{8\pi z \sin \pi ir_1} \int_0^\infty (u^{1+2ir_1} J_{1+2ir_1}(u))' f(u) du.$$

Integration by parts gives

$$G_1(z, y) = \frac{c^2}{8\pi z \sin \pi ir_1} \int_0^\infty u^{1+2ir_1} J_{1+2ir_1}(u) f'(u) du \\ = -\frac{c^2}{8\pi z \sin \pi ir_1} \int_0^\infty u^{2+2ir_1} J_{2+2ir_1}(u) \left(\frac{1}{u} f'(u) \right)' du.$$

Repeating the procedure n times, we have

$$\begin{aligned} G_1(z, y) &= (-1)^{n+1} \frac{c^2}{8\pi z \sin \pi i r_1} \int_0^\infty u^{n+2ir_1} J_{n+2ir_1}(u) h_n(u) du = \\ &= (-1)^{n+1} \frac{c^2}{8\pi z \sin \pi i r_1} \int_{u \sim \frac{\sqrt{Mz}}{c}} \frac{J_{n+2ir_1}(u)}{u^{n-1-2ir_1}} u^{2n-1} h_n(u) du, \end{aligned}$$

where

$$h_0(u) = f(u),$$

$$h_1(u) = f'(u),$$

$$h_n(u) = (u^{-1} h_{n-1}(u))' \text{ for } n \geq 2.$$

By induction for $n \geq 1$

$$u^{2n-1} h_n(u) = \sum_{i=0}^n c(i, n) f^{(i)}(u) u^i$$

with

$$\begin{aligned} f^{(i)}(u) u^i &\ll \sum_{j+l+m=i} (g_1^{(j)}(u) u^j) (g_2^{(l)}(u) u^l) u^{2ir_1} \\ &\ll \sum_{j+m < i} (g_1^{(j)}(u) u^j) (g_2^{(i-j-m)}(u) u^{i-j-m}). \end{aligned}$$

Faà di Bruno's formula and the estimate (A.4) give

$$\begin{aligned} u^j g_1^{(j)}(u) &= u^j \frac{\partial^j}{\partial u^j} \left(J_{k-1} \left(u \sqrt{\frac{y}{z}} \right) \right) = \left(\sqrt{\frac{y}{z}} u \right)^j J_{k-1}^{(j)} \left(u \sqrt{\frac{y}{z}} \right) \\ &\ll \frac{(u \sqrt{\frac{y}{z}})^{k-1} (1 + u \sqrt{\frac{y}{z}})^j}{(1 + u \sqrt{\frac{y}{z}})^{k-1/2}}. \end{aligned}$$

Applying Faà di Bruno's formula to the second function, we obtain

$$\begin{aligned} u^{i-j-m} g_2^{(i-j-m)}(u) &= u^{i-j-m} \frac{\partial^{i-j-m}}{\partial u^{i-j-m}} F_{M,N} \left(\frac{c^2 u^2}{16\pi^2 z}, y \right) \\ &= u^{i-j-m} \sum_{\substack{(m_1, m_2) \\ m_1 + 2m_2 = i-j-m}} \frac{(i-j-m)!}{m_1! m_2! (2!)^{m_2}} \\ &\quad \times F_{M,N}^{(m_1+m_2)} \left(\frac{c^2 u^2}{16\pi^2 z}, y \right) \left(\frac{2c^2}{16\pi^2 z} u \right)^{m_1} \left(\frac{2c^2}{16\pi^2 z} \right)^{m_2}. \end{aligned}$$

Lemma 3.6 implies

$$u^{i-j-m} g_2^{(i-j-m)}(u) \ll \sum_{\substack{(m_1, m_2) \\ m_1 + 2m_2 = i-j-m}} \left(\frac{c^2}{16\pi^2 z} u^2 \right)^{m_1 + m_2} \\ \times F_{M,N}^{(m_1 + m_2)} \left(\frac{c^2 u^2}{16\pi^2 z}, y \right) \ll (MN)^{-1/2}.$$

And for the J -Bessel function we use the trivial bound $J_{n+2ir_1}(u) \ll 1$. Then

$$G_1(z, y) \ll \left(\frac{Qc}{\sqrt{Mz}} \right)^n \frac{M^{1/2}}{N^{1/2}} \left(\frac{\sqrt{MN}}{c} \right)^{k-1} Q^{-k+1/2}$$

for every integer $n > 0$. The same bound is valid for $G_{r_1}^-(z, y)$. So, if $z > Z$, the value of $G_{r_1}^-(z, y)$ is small.

Suppose $z \leq Z$. One can estimate $G_{r_1}^-(z, y)$ directly (without integration by parts)

$$G_{r_1}^-(z, y) \ll \frac{M^{1/2}}{N^{1/2}} \left(\frac{\sqrt{MN}}{c} \right)^{k-1} Q^{-k+1/2}.$$

Since $y \in [N/2, 3N]$, we can add a multiple $(1 + \frac{y}{Y})^{-n_2}$.

Combining two estimates for $G_{r_1}^-(z, y)$ in one, we have that for all positive n_1 and n_2

$$G_{r_1}^-(z, y) \ll \left(1 + \frac{z}{Z}\right)^{-n_1} \left(1 + \frac{y}{Y}\right)^{-n_2} \frac{M^{1/2}}{N^{1/2}} \left(\frac{\sqrt{MN}}{c} \right)^{k-1} Q^{-k+1/2}.$$

Analogously, using relation (A.3) and bound for the K -Bessel function (A.6), we may estimate $G_{r_1}^+(z, y)$. Finally, differentiating $G_{r_1}^\pm(z, y)$ in z variable j times and in y variable i times, we find

$$z^j y^i \frac{\partial^j}{\partial z^j} \frac{\partial^i}{\partial y^i} G_{r_1}^\pm(z, y) \ll \left(1 + \frac{z}{Z}\right)^{-n_1} \left(1 + \frac{y}{Y}\right)^{-n_2} \\ \times \frac{M^{1/2}}{N^{1/2}} \left(\frac{\sqrt{MN}}{c} \right)^{k-1} Q^{j+i-k+1/2}$$

for all positive n_1 and n_2 . An extra multiple of Q^{i+j} is obtained by differentiating the Bessel functions under the integral. Indeed, by Faà di Bruno's formula

$$z^j \frac{\partial^j}{\partial z^j} J_{2ir_1}(\alpha\sqrt{z}) = z^j \sum \binom{j}{m_1, m_2, \dots, m_j} \\ \times J_{2ir_1}^{(m_1 + m_2 + \dots + m_j)}(\alpha\sqrt{z}) \cdot \prod_{n=1}^j \left(\frac{\alpha z^{1/2-n}}{n!} \right)^{m_n},$$

where

$$\alpha := \frac{4\pi\sqrt{x}}{c}$$

and the sum is over all j -tuples (m_1, m_2, \dots, m_j) such that $1 \cdot m_1 + 2 \cdot m_2 + \dots + j \cdot m_j = j$. Formula (A.9) gives

$$J_{2ir_1}^{(b)}(z) = \frac{1}{2^b} \sum_{t=0}^b (-1)^t \binom{b}{t} J_{2ir_1-b+2t}(z).$$

When $z > Z$, the maximum of $z^j \frac{\partial^j}{\partial z^j} J_{2ir_1}(\alpha\sqrt{z})$ is attained when $m_1 + m_2 + \dots + m_j = j$. Therefore,

$$z^j \frac{\partial^j}{\partial z^j} J_{2ir_1}(\alpha\sqrt{z}) \ll (\alpha\sqrt{z})^j \sum_{t=0}^j J_{2ir_1-j+2t}(\alpha\sqrt{z}).$$

This gives an extra multiple $\left(\frac{\sqrt{Mz}}{c}\right)^j$ and

$$G_{r_1}^-(z, y) \ll \left(\frac{Qc}{\sqrt{Mz}}\right)^{n-j} Q^j \frac{M^{1/2}}{N^{1/2}} \left(\frac{\sqrt{MN}}{c}\right)^{k-1} Q^{-k+1/2}$$

for every integer $n > 0$.

In the similar manner

$$y^i \frac{\partial^i}{\partial y^i} [J_{k-1}(\alpha\sqrt{y}) F_{M,N}(x, y)] \ll \sum_{a=0}^i y^a (J_{k-1}(\alpha\sqrt{y}))^{(a)} y^{i-a} F_{M,N}(x, y)^{(i-a)}$$

gives an extra factor of Q^i . \square

5.2 Applying theorem 2.9

According to the formula (3.32), the off-off-diagonal term is equal to

$$(5.4) \quad M^{OOD} = M^{OOD}(0) - \tau_{1/2+ir_2}(p) M^{OOD}(1) + M^{OOD}(2),$$

where for $B = 0, 1, 2$

$$M^{OOD}(B) = \frac{2\pi i^{-k}}{\hat{q}^{2t_1+2t_2}} \left(\sum_{M, N \ll q^{1+\epsilon}} \sum_{\substack{q|c \\ c \ll C}} \frac{1}{c^2} \sum_{h \neq 0} S(0, h, c) (T_h^-(c, B) + T_h^+(c, B)) \right. \\ \left. - \frac{1}{p} \sum_{M, N \ll q^{1+\epsilon}} \sum_{\substack{\frac{q}{p}|c \\ c \ll C}} \frac{1}{c^2} \sum_{h \neq 0} S(0, h, c) (T_h^-(c, B) + T_h^+(c, B)) \right).$$

Since k is even, $i^{-k} = i^k$.

Lemma 5.3. *Up to the error*

$$O_{\mathbf{r},\epsilon} \left(q^\epsilon \left(q^{-\frac{k-1-2\theta}{8-8\theta}} + q^{-1/4} \right) \right),$$

we have

$$(5.5) \quad T_h^\mp(c, B) = \pm \int_0^\infty \delta_{h \pm p^B y > 0} G_{r_1}^\mp(h \pm p^B y, p^B y) \Lambda(h \pm p^B y, p^B y) dy$$

with

$$(5.6) \quad \Lambda(h \pm p^B y, p^B y) := \sum_{w=1}^\infty S(0, h, w) \sum_{\epsilon_1, \epsilon_2 = \pm 1} \frac{(p^B, w)^{1+2i\epsilon_2 r_2}}{w^{2+2i\epsilon_1 r_1 + 2i\epsilon_2 r_2}} \\ \times \zeta(1 + 2i\epsilon_1 r_1) \zeta(1 + 2i\epsilon_2 r_2) (h \pm p^B y)^{i\epsilon_1 r_1} y^{i\epsilon_2 r_2}.$$

Proof. We apply theorem 2.9 to the function $T_h^\mp(c, B)$ and let $x = h \pm p^B y$. Then

$$T_h^\mp(c, B) = \pm \int_0^\infty \delta_{h \pm p^B y > 0} G_{r_1}^\mp(h \pm p^B y, p^B y) \Lambda(h \pm p^B y, p^B y) dy + O(ET),$$

where

$$\Lambda(h \pm p^B y, p^B y) := \sum_{w=1}^\infty S(0, h, w) \sum_{\epsilon_1, \epsilon_2 = \pm 1} \frac{(p^B, w)^{1+2i\epsilon_2 r_2}}{w^{2+2i\epsilon_1 r_1 + 2i\epsilon_2 r_2}} \\ \times \zeta(1 + 2i\epsilon_1 r_1) \zeta(1 + 2i\epsilon_2 r_2) (h \pm p^B y)^{i\epsilon_1 r_1} y^{i\epsilon_2 r_2}$$

and the error term is

$$ET := \frac{M^{1/2}}{N^{1/2}} \left(\frac{\sqrt{MN}}{c} \right)^{k-1} Q^{-k+1/2} Q^{5/4} (Z + N)^{1/4} (ZN)^{1/4+\epsilon}.$$

Since $Z = Q^2 \frac{c^2}{M} > N$,

$$ET \ll M^{1/2} N^{1/4} \left(\frac{\sqrt{MN}}{c} \right)^{k-2} Q^{-k+11/4}.$$

Note that $T_h^\mp(c)$ is small when $|h| \gg Zq^\epsilon$ because G^\mp is small when $z \gg Zq^\epsilon$. This allows us to add $\left(1 + \frac{|h|}{Z}\right)^{-2}$ into the error term ET . Multiplying by $S(0, h, c)$ and summing over h , we have

$$ET_1 := \sum_{h \neq 0} S(0, h, c) \left(1 + \frac{|h|}{Z}\right)^{-2} ET \\ \ll c^2 \frac{N^{1/4}}{M^{1/2}} \left(\frac{\sqrt{MN}}{c} \right)^{k-2} Q^{-k+2+11/4}.$$

Finally, we sum over c . If $k = 2$

$$\begin{aligned} \sum_{\substack{c \leq C \\ q|c}} c^{-2} ET_1 &\ll q^\epsilon \frac{N^{1/4}}{M^{1/2}} \sum_{\substack{c \leq C \\ q|c}} \left[1 + \left(\frac{\sqrt{MN}}{c} \right)^{11/4} \right] \\ &\ll q^\epsilon \left(\frac{N^{1/4}}{M^{1/2}} \frac{C}{q} + \frac{N^{13/8} M^{7/8}}{q^{11/4}} \right). \end{aligned}$$

The optimal value of C can be found by making equal the first summand and the error term in lemma 3.9

$$\frac{N^{1/4}}{M^{1/2}} \frac{C}{q} = \left(\frac{\sqrt{MN}}{C} \right)^{1-2\theta}.$$

Thus, $C := \min \left(q^{\frac{1}{2-2\theta}} M^{1/2} N^{\frac{1-4\theta}{8-8\theta}}, q^{\frac{9-8\theta}{8-8\theta}} \right)$ and

$$\sum_{M, N \ll q^{1+\epsilon}} \sum_{\substack{c \leq C \\ q|c}} c^{-2} ET_1 \ll q^\epsilon (q^{-\frac{1-2\theta}{8-8\theta}} + q^{-1/4}).$$

If $k \geq 4$, then

$$\begin{aligned} \sum_{\substack{c \leq C \\ q|c}} c^{-2} ET_1 &\ll q^\epsilon \frac{N^{1/4}}{M^{1/2}} \sum_{\substack{c \leq C \\ q|c}} \left[\left(\frac{\sqrt{MN}}{c} \right)^{k-2} + \left(\frac{\sqrt{MN}}{c} \right)^{11/4} \right] \\ &\ll q^\epsilon \left(\frac{N^{1/4}}{M^{1/2}} \left(\frac{\sqrt{MN}}{q} \right)^{k-2} + \frac{N^{13/8} M^{7/8}}{q^{11/4}} \right). \end{aligned}$$

Therefore,

$$\sum_{M, N \ll q^{1+\epsilon}} \sum_{\substack{c \leq C \\ q|c}} c^{-2} ET_1 \ll q^{\epsilon-1/4}.$$

Combining two estimates in one, we have that for any even k

$$\sum_{M, N \ll q^{1+\epsilon}} \sum_{\substack{c \leq C \\ q|c}} c^{-2} ET_1 \ll q^\epsilon \left(q^{-\frac{k-1-2\theta}{8-8\theta}} + q^{-1/4} \right).$$

□

5.3 Extension of summations

Analogously to the off-diagonal term, at the cost of admissible error, we can reintroduce summation over $\max(M, N) \geq q^{1+\epsilon}$ and extend the summation over c up to some large value $C_{max} = q^\Omega$.

Proposition 5.4. *For $l = 0, 1$, we have*

$$(5.7) \quad \sum_{\max(M, N) \ll q^{1+\epsilon}} \sum_{\substack{\frac{q}{p^l} | c \\ C < c \leq C_{\max}}} \frac{1}{c^2} \sum_{h \neq 0} S(0, h, c) T_h^\pm(c, B) \ll_{\epsilon, r} q^{\epsilon - \frac{k-1}{8-8\theta}}.$$

Proof. Consider $T_h^\pm(c, B)$ given by equation (5.5). We split the sum over w in expression (5.6) into two parts: $w < q$ and $w \geq q$. If $w < q$ we follow section 10 of [DFI3] to show that

$$\sum_{\substack{\frac{q}{p^l} | c \\ C < c \leq C_{\max}}} \frac{1}{c^2} \sum_{h \neq 0} S(0, h, c) T_h^\pm(c, B) \ll_{\epsilon, r} q^\epsilon \frac{(\sqrt{MN})^k}{q C^{k-1}}.$$

If $w \geq q$ we estimate the absolute value of $T_h^\pm(c, B)$ using

$$G_{r_1}^\pm(h \pm p^B y, p^B y) \ll \left(1 + \frac{M}{c^2} (h \pm p^B y)\right)^{-2} \frac{M^{1/2}}{N^{1/2}} \left(\frac{\sqrt{MN}}{c}\right)^{k-1}$$

and

$$S(0, w, c) \ll (w, c).$$

This gives

$$\sum_{\substack{\frac{q}{p^l} | c \\ C < c \leq C_{\max}}} \frac{1}{c^2} \sum_{h \neq 0} S(0, h, c) T_h^\pm(c, B) \ll_{\epsilon, r} q^\epsilon \frac{(\sqrt{MN})^k}{q C^{k-1}} \frac{C}{qM}.$$

Finally, taking $C = \min\left(q^{\frac{1}{2-2\theta}} M^{1/2} N^{\frac{1-4\theta}{8-8\theta}}, q^{\frac{9-8\theta}{8-8\theta}}\right)$ and performing dyadic summation over M, N , we obtain the assertion. \square

Proposition 5.5. *For any $\epsilon > 0$, any $A > 0$ and $l = 0, 1$*

$$(5.8) \quad \sum_{\substack{\frac{q}{p^l} | c}} \frac{1}{c^2} \sum_{\max(M, N) \gg q^{1+\epsilon}} \sum_{h \neq 0} S(0, h, c) T_h^\pm(c, B) \ll_{\epsilon, A, r} q^{-A}.$$

Proof. The assertion follows from the rapid decay of $F_{M, N}$. See the proof of proposition 3.7 for details. \square

Now it is possible to combine all functions F_M into F and replace $\sum_{M, N} F_{M, N}$ by

$$(5.9) \quad F(x, y) := \frac{1}{\sqrt{xy}} W_{t_1, r_1}\left(\frac{x}{\hat{q}^2}\right) W_{t_2, r_2}\left(\frac{y}{\hat{q}^2}\right) F(x) F(y),$$

where $F(x)$ is a smooth function, compactly supported in $[1/2, \infty)$ such that $F(x) = 1$ for $x \geq 1$.

Lemma 5.6. *One has*

$$\begin{aligned}
M^{OOD}(B) &= 2\pi i^k \hat{q}^{-2t_1-2t_2} \sum_{\epsilon_1, \epsilon_2 = \pm 1} \zeta(1 + 2i\epsilon_1 r_1) \zeta(1 + 2i\epsilon_2 r_2) \\
&\times \left(\sum_{g,v} \frac{\mu(g)}{g^2} \frac{\mu(v)}{v^{2+2i\epsilon_1 r_1+2i\epsilon_2 r_2}} \sum_{\substack{q|cg \\ cg < q^\Omega}} \frac{1}{c} \sum_w \frac{(p^B, wv)^{1+2i\epsilon_2 r_2}}{w^{1+2i\epsilon_1 r_1+2i\epsilon_2 r_2}} \sum_{\substack{h \neq 0 \\ [w,c]|h}} V(h) \right. \\
&\quad \left. - \frac{1}{p} \sum_{g,v} \frac{\mu(g)}{g^2} \frac{\mu(v)}{v^{2+2i\epsilon_1 r_1+2i\epsilon_2 r_2}} \sum_{\substack{\frac{q}{p}|cg \\ cg < q^\Omega}} \frac{1}{c} \sum_w \frac{(p^B, wv)^{1+2i\epsilon_2 r_2}}{w^{1+2i\epsilon_1 r_1+2i\epsilon_2 r_2}} \sum_{\substack{h \neq 0 \\ [w,c]|h}} V(h) \right),
\end{aligned}$$

where

$$\begin{aligned}
V(h) &= -\frac{1}{(2\pi i)^2} \frac{1}{p^{B(1+i\epsilon_2 r_2)}} \int_{\Re \beta = 0.7} \int_{\Re z = -0.1} \frac{\Gamma(\beta + ir_1) \Gamma(\beta - ir_1)}{\Gamma(1+z) \Gamma(k+z)} \\
&\quad \times \frac{(4\pi)^{k+2z-2\beta} 2^{-k-2z+2\beta}}{\sin(\pi z)} (cg)^{-k+1-2z+2\beta} h^{k/2+z-\beta+i\epsilon_1 r_1+i\epsilon_2 r_2} \\
&\times \int_{x=0}^{\infty} x^{z-\beta+k/2} W_{t_1, r_1} \left(\frac{x}{\hat{q}^2} \right) F(x) \frac{dx}{x} \int_{y=0}^{\infty} y^{z+k/2+i\epsilon_2 r_2} W_{t_2, r_2} \left(\frac{hy}{\hat{q}^2} \right) F(hy) \\
&\times \left(\frac{\cos(\pi\beta)}{(1+y)^{\beta-i\epsilon_1 r_1}} + \delta_{y>1} \frac{\cos(\pi\beta)}{(-1+y)^{\beta-i\epsilon_1 r_1}} + \delta_{y<1} \frac{\cos(\pi ir_1)}{(1-y)^{\beta-i\epsilon_1 r_1}} \right) \frac{dy}{y} dz d\beta.
\end{aligned}$$

Proof. Lemma 5.3 yields

$$\begin{aligned}
T_h^-(c, B) + T_h^+(c, B) &= \sum_{w=1}^{\infty} S(0, h, w) \sum_{\epsilon_1, \epsilon_2 = \pm 1} \frac{(p^B, w)^{1+2i\epsilon_2 r_2}}{w^{2+2i\epsilon_1 r_1+2i\epsilon_2 r_2}} \zeta(1 + 2i\epsilon_1 r_1) \\
&\times \zeta(1 + 2i\epsilon_2 r_2) \int_0^{\infty} \left[\delta_{h+p^B y > 0} G_{r_1}^-(h + p^B y, p^B y) (h + p^B y)^{i\epsilon_1 r_1} y^{i\epsilon_2 r_2} \right. \\
&\quad \left. + \delta_{h-p^B y > 0} G_{r_1}^+(h - p^B y, p^B y) (h - p^B y)^{i\epsilon_1 r_1} y^{i\epsilon_2 r_2} \right] dy.
\end{aligned}$$

We plug in the expressions for $G_{r_1}^-$ and $G_{r_1}^+$ given by (3.27) and (3.28) and use the identity

$$F(x, p^B y) = \frac{1}{(p^B x y)^{1/2}} W_{t_1, r_1} \left(\frac{x}{\hat{q}^2} \right) W_{t_2, r_2} \left(\frac{p^B y}{\hat{q}^2} \right) F(x) F(p^B y).$$

This gives

$$\begin{aligned}
T_h^-(c, B) + T_h^+(c, B) &= \sum_{w=1}^{\infty} S(0, h, w) \sum_{\epsilon_1, \epsilon_2 = \pm 1} \frac{(p^B, w)^{1+2i\epsilon_2 r_2}}{w^{2+2i\epsilon_1 r_1 + 2i\epsilon_2 r_2}} \\
&\quad \times \zeta(1 + 2i\epsilon_1 r_1) \zeta(1 + 2i\epsilon_2 r_2) \\
&\quad \times 2\pi \int_0^\infty \int_0^\infty \frac{1}{(p^B x y)^{1/2}} W_{t_1, r_1} \left(\frac{x}{\hat{q}^2} \right) W_{t_2, r_2} \left(\frac{p^B y}{\hat{q}^2} \right) J_{k-1} \left(\frac{4\pi \sqrt{x p^B y}}{c} \right) \\
&\quad \times \left[\delta_{h+p^B y > 0} k_0 \left(\frac{4\pi \sqrt{x(h+p^B y)}}{c}, 1/2 + ir_1 \right) (h+p^B y)^{i\epsilon_1 r_1} y^{i\epsilon_2 r_2} \right. \\
&\quad \left. + \delta_{h-p^B y > 0} k_1 \left(\frac{4\pi \sqrt{x(h-p^B y)}}{c}, 1/2 + ir_1 \right) (h-p^B y)^{i\epsilon_1 r_1} y^{i\epsilon_2 r_2} \right] \\
&\quad \times F(x) F(p^B y) dx dy.
\end{aligned}$$

The off-off-diagonal term

$$\begin{aligned}
M^{OOD}(B) &= 2\pi i^k \hat{q}^{-2t_1-2t_2} \left(\sum_{q|c} \frac{1}{c^2} \sum_{h \neq 0} S(0, h, c) (T_h^-(c, B) + T_h^+(c, B)) \right. \\
&\quad \left. - \frac{1}{p} \sum_{\frac{q}{p}|c} \frac{1}{c^2} \sum_{h \neq 0} S(0, h, c) (T_h^-(c, B) + T_h^+(c, B)) \right)
\end{aligned}$$

contains two Ramanujan sums $S(0, h, c)$ and $S(0, h, w)$. Applying the formulas

$$S(0, h, c) = \sum_{\substack{gc_1=c \\ c_1|h}} \mu(g) c_1, \quad S(0, h, w) = \sum_{\substack{vw_1=c \\ w_1|h}} \mu(v) w_1,$$

we obtain

$$\begin{aligned}
M^{OOD}(B) &= 2\pi i^k \hat{q}^{-2t_1-2t_2} \sum_{\epsilon_1, \epsilon_2 = \pm 1} \zeta(1 + 2i\epsilon_1 r_1) \zeta(1 + 2i\epsilon_2 r_2) \\
&\quad \times \left(\sum_{g, v} \frac{\mu(g)}{g^2} \frac{\mu(v)}{v^{2+2i\epsilon_1 r_1 + 2i\epsilon_2 r_2}} \sum_{\substack{q|cg \\ cg < q^\Omega}} \frac{1}{c} \sum_w \frac{(p^B, wv)^{1+2i\epsilon_2 r_2}}{w^{1+2i\epsilon_1 r_1 + 2i\epsilon_2 r_2}} \sum_{\substack{h \neq 0 \\ [w, c]|h}} V(h) \right. \\
&\quad \left. - \frac{1}{p} \sum_{g, v} \frac{\mu(g)}{g^2} \frac{\mu(v)}{v^{2+2i\epsilon_1 r_1 + 2i\epsilon_2 r_2}} \sum_{\substack{\frac{q}{p}|cg \\ cg < q^\Omega}} \frac{1}{c} \sum_w \frac{(p^B, wv)^{1+2i\epsilon_2 r_2}}{w^{1+2i\epsilon_1 r_1 + 2i\epsilon_2 r_2}} \sum_{\substack{h \neq 0 \\ [w, c]|h}} V(h) \right),
\end{aligned}$$

where

$$\begin{aligned}
V(h) &= 2\pi \int_0^\infty \int_0^\infty \frac{1}{(p^B xy)^{1/2}} W_{t_1, r_1} \left(\frac{x}{\hat{q}^2} \right) W_{t_2, r_2} \left(\frac{p^B y}{\hat{q}^2} \right) \\
&\times \left[\delta_{h+p^B y > 0} k_0 \left(\frac{4\pi \sqrt{x(h+p^B y)}}{cg}, 1/2 + ir_1 \right) (h+p^B y)^{i\epsilon_1 r_1} y^{i\epsilon_2 r_2} \right. \\
&\quad \left. + \delta_{h-p^B y > 0} k_1 \left(\frac{4\pi \sqrt{x(h-p^B y)}}{cg}, 1/2 + ir_1 \right) (h-p^B y)^{i\epsilon_1 r_1} y^{i\epsilon_2 r_2} \right] \\
&\quad \times J_{k-1} \left(\frac{4\pi \sqrt{x p^B y}}{cg} \right) F(x) F(p^B y) dx dy.
\end{aligned}$$

In the expression $V(h)$ we replace negative h by their absolute value and make the change of variables $\frac{p^B y}{h} \rightarrow y$ in the integral. As a result,

$$\begin{aligned}
V(h) &= 2\pi \frac{h^{1/2+i\epsilon_1 r_1+i\epsilon_2 r_2}}{p^{B(1+i\epsilon_2 r_2)}} \int_0^\infty \int_0^\infty \frac{y^{i\epsilon_2 r_2}}{(xy)^{1/2}} W_{t_1, r_1} \left(\frac{x}{\hat{q}^2} \right) W_{t_2, r_2} \left(\frac{hy}{\hat{q}^2} \right) \\
&\times \left[(1+y)^{i\epsilon_1 r_1} k_0 \left(\frac{4\pi \sqrt{xh(1+y)}}{cg}, 1/2 + ir_1 \right) + \delta_{y>1} (-1+y)^{i\epsilon_1 r_1} \right. \\
&\quad \times k_0 \left(\frac{4\pi \sqrt{xh(y-1)}}{cg}, 1/2 + ir_1 \right) + \delta_{y<1} (1-y)^{i\epsilon_1 r_1} \\
&\quad \left. \times k_1 \left(\frac{4\pi \sqrt{xh(1-y)}}{cg}, 1/2 + ir_1 \right) \right] J_{k-1} \left(\frac{4\pi \sqrt{xhy}}{cg} \right) F(x) F(hy) dx dy.
\end{aligned}$$

Finally, we use Mellin transforms of Bessel functions (B.5), (B.9) and (B.10), so that

$$\begin{aligned}
V(h) &= -\frac{p^{-B(1+i\epsilon_2 r_2)}}{(2\pi i)^2} \int_{\Re \beta = 0.7} \int_{\Re z = -0.1} \frac{\Gamma(\beta + ir_1) \Gamma(\beta - ir_1)}{\Gamma(1+z) \Gamma(k+z)} \\
&\times \int_{\Re \beta = 0.7} \int_{\Re z = -0.1} \frac{(4\pi)^{k+2z-2\beta} 2^{2\beta-k-2z}}{\sin(\pi z)} (cg)^{-k+1-2z+2\beta} h^{k/2+z-\beta+i\epsilon_1 r_1+i\epsilon_2 r_2} \\
&\times \int_{x=0}^\infty x^{z-\beta+k/2} W_{t_1, r_1} \left(\frac{x}{\hat{q}^2} \right) F(x) \frac{dx}{x} \int_{y=0}^\infty y^{z+k/2+i\epsilon_2 r_2} W_{t_2, r_2} \left(\frac{hy}{\hat{q}^2} \right) F(hy) \\
&\times \left(\frac{\cos(\pi\beta)}{(1+y)^{\beta-i\epsilon_1 r_1}} + \delta_{y>1} \frac{\cos(\pi\beta)}{(-1+y)^{\beta-i\epsilon_1 r_1}} + \delta_{y<1} \frac{\cos(\pi ir_1)}{(1-y)^{\beta-i\epsilon_1 r_1}} \right) \frac{dy}{y} dz d\beta.
\end{aligned}$$

Note that we shifted the contour of integration given in (B.10) to $\Re \beta = 0.7$, which is possible due to the rapid decay of the x integral in β . The change of the order of integration in $V(h)$ is justified by absolute convergence of all integrals. \square

5.4 Replacing $F(x)F(hy)$ by 1 on the interval $[0, \infty)^2$

This step allows us to simplify the integration and can be performed with a cost of negligible error.

5.4.1 y-integral

Consider

$$(5.10) \quad IY := \int_{y=0}^{\infty} y^{z+k/2+i\epsilon_2 r_2} W_{t_2, r_2} \left(\frac{hy}{\hat{q}^2} \right) F(hy) \\ \times \left(\frac{\cos(\pi\beta)}{(1+y)^{\beta-i\epsilon_1 r_1}} + \delta_{y>1} \frac{\cos(\pi\beta)}{(-1+y)^{\beta-i\epsilon_1 r_1}} + \delta_{y<1} \frac{\cos(\pi i r_1)}{(1-y)^{\beta-i\epsilon_1 r_1}} \right) \frac{dy}{y}.$$

Lemma 5.7. *The function $F(hy)$ can be replaced by 1 in IY with an error*

$$O_{\epsilon}(q^{-1/2+\epsilon}).$$

Proof. $F(hy)$ is a smooth function, compactly supported in $[1/2, \infty)$ such that $F(hy) = 1$ for $hy \geq 1$. Thus, we only need to estimate the integral for $y < 1/h$. It is bounded by $(\frac{1}{h})^{k/2+\Re z} \cos \pi\beta$. We are left to estimate

$$T := \sum_{g,v,w} \frac{1}{g^2 v^2 w} \sum_{\substack{q|cg \\ cg < q^{\Omega}}} \frac{1}{c} \sum_{[c,w]|h} h^{-\beta} \int_{\Re \beta = 0.7} \int_{\Re z = -0.1} \frac{\Gamma(\beta + i r_1) \Gamma(\beta - i r_1)}{\Gamma(1+z) \Gamma(k+z)} \\ \times \frac{\cos \pi \beta}{\sin(\pi z)} (cg)^{-k+1-2z+2\beta} \int_{x=0}^{\infty} x^{z-\beta+k/2} W_{t_1, r_1} \left(\frac{x}{\hat{q}^2} \right) F(x) \frac{dx}{x}.$$

To make the sums over h and w absolutely convergent, one has to move β contour to the right $\Re \beta > 1$. At the same time, partial integration shows that the x -integral decays rapidly in β :

$$\int_0^{\infty} x^{z-\beta+k/2} W_{t_1, r_1} \left(\frac{x}{\hat{q}^2} \right) F(x) \frac{dx}{x} \\ = \frac{1}{(z-\beta+k/2)(z-\beta+k/2)\dots(z-\beta+k/2+n-1)} \\ \times \int_0^{\infty} \frac{\partial^n}{\partial x^n} \left(W_{t_1, r_1} \left(\frac{x}{\hat{q}^2} \right) F(x) \right) x^{z-\beta+k/2+n-1} dx \ll \frac{1}{|\beta|^n} q^{z-\beta+k/2}.$$

Assume that $\Re \beta > 1$. We have

$$T \ll q^{z-\beta+k/2} \sum_{v,w,h} \frac{1}{v^2 w^{1+\beta} h^{\beta}} \sum_{\substack{q|cg \\ cg < q^{\Omega}}} \frac{1}{c^{1+\beta} g^2} (cg)^{-k+1-2z+2\beta} \\ \ll q^{z-\beta+k/2} \sum_{\substack{q|cg \\ cg < q^{\Omega}}} (cg)^{-k-1-2z+2\beta} \ll q^{z-\beta+k/2-1} q^{\Omega(-k-2z+2\beta)}.$$

Moving β contour to $\Re\beta = k/2 + \delta$ and z contour to $-\delta$, M^{OOD} is dominated by

$$q^{-1} q^{4\delta\Omega - 2\delta}.$$

Choosing $\delta = \frac{1+2\epsilon}{4(2\Omega-1)}$, we obtain the result. \square

Lemma 5.8. *One has*

$$(5.11) \quad IY = \frac{1}{2\pi i} \int_{\Re t = k/2 - 0.2} \frac{P_r(t)}{P_r(t_2)} \frac{\Gamma(t + ir_2 + k/2)\Gamma(t - ir_2 + k/2)}{\Gamma(t_2 + ir_2 + k/2)\Gamma(t_2 - ir_2 + k/2)} \\ \times \left(\frac{\hat{q}^2}{h}\right)^t \frac{\Gamma(k/2 + z - t + i\epsilon_2 r_2)\Gamma(-k/2 - z + t + \beta - i\epsilon_1 r_1 - i\epsilon_2 r_2)}{\Gamma(\beta - i\epsilon_1 r_1)} \\ \times \zeta_q(1 + 2t) \left(\cos(\pi\beta) + \frac{\cos(\pi\beta) \sin(\pi(k/2 + z - t + i\epsilon_2 r_2))}{\sin(\pi(\beta - i\epsilon_1 r_1))} \right. \\ \left. + \frac{\cos(\pi i r_1) \sin(\pi(-k/2 - z + t + \beta - i\epsilon_1 r_1 - i\epsilon_2 r_2))}{\sin(\pi(\beta - i\epsilon_1 r_1))} \right) \frac{2tdt}{t^2 - t_2^2}.$$

Proof. By lemma 5.7, the y -integral is equal to

$$IY = \int_{y=0}^{\infty} y^{z+k/2+i\epsilon_2 r_2} W_{t_2, r_2} \left(\frac{hy}{\hat{q}^2} \right) \\ \times \left(\frac{\cos(\pi\beta)}{(1+y)^{\beta-i\epsilon_1 r_1}} + \delta_{y>1} \frac{\cos(\pi\beta)}{(-1+y)^{\beta-i\epsilon_1 r_1}} + \delta_{y<1} \frac{\cos(\pi i r_1)}{(1-y)^{\beta-i\epsilon_1 r_1}} \right) \frac{dy}{y}.$$

We plug in the expression

$$W_{t_2, r_2} \left(\frac{hy}{\hat{q}^2} \right) = \frac{1}{2\pi i} \int_{\Re t = k/2 - 0.2} \frac{P_r(t)}{P_r(t_2)} \zeta_q(1 + 2t) \\ \times \frac{\Gamma(t + ir_2 + k/2)\Gamma(t - ir_2 + k/2)}{\Gamma(t_2 + ir_2 + k/2)\Gamma(t_2 - ir_2 + k/2)} \left(\frac{hy}{\hat{q}^2} \right)^{-t} \frac{2tdt}{t^2 - t_2^2}.$$

Note that we shifted $\Re t$ from 3 to $k/2 - 0.2$ without crossing any poles. This step is required to ensure that all poles of $\Gamma(-k/2 - z + t + \beta - i\epsilon_1 r_1 - i\epsilon_2 r_2)$ lie to the left of the t contour. Therefore,

$$IY = \frac{1}{2\pi i} \int_{\Re t = k/2 - 0.2} \frac{P_r(t)}{P_r(t_2)} \frac{\Gamma(t + ir_2 + k/2)\Gamma(t - ir_2 + k/2)}{\Gamma(t_2 + ir_2 + k/2)\Gamma(t_2 - ir_2 + k/2)} \zeta_q(1 + 2t) \\ \times \left(\frac{\hat{q}^2}{h} \right)^t \int_{y=0}^{\infty} y^{z+k/2+i\epsilon_2 r_2-t} \left(\frac{\cos(\pi\beta)}{(1+y)^{\beta-i\epsilon_1 r_1}} + \delta_{y>1} \frac{\cos(\pi\beta)}{(-1+y)^{\beta-i\epsilon_1 r_1}} \right. \\ \left. + \delta_{y<1} \frac{\cos(\pi i r_1)}{(1-y)^{\beta-i\epsilon_1 r_1}} \right) \frac{dy}{y} \frac{2tdt}{t^2 - t_2^2}.$$

Mellin transforms (B.1), (B.2) and (B.3) and Euler's reflection formula give

$$\begin{aligned}
IY = & \frac{1}{2\pi i} \int_{\Re t = k/2 - 0.2} \frac{P_r(t)}{P_r(t_2)} \frac{\Gamma(t + ir_2 + k/2)\Gamma(t - ir_2 + k/2)}{\Gamma(t_2 + ir_2 + k/2)\Gamma(t_2 - ir_2 + k/2)} \zeta_q(1 + 2t) \\
& \times \left(\frac{\hat{q}^2}{h} \right)^t \left(\cos(\pi\beta) + \frac{\cos(\pi\beta) \sin(\pi(k/2 + z - t + i\epsilon_2 r_2))}{\sin(\pi(\beta - i\epsilon_1 r_1))} \right. \\
& \quad \left. + \frac{\cos(\pi i r_1) \sin(\pi(-k/2 - z + t + \beta - i\epsilon_1 r_1 - i\epsilon_2 r_2))}{\sin(\pi(\beta - i\epsilon_1 r_1))} \right) \\
& \times \frac{\Gamma(k/2 + z - t + i\epsilon_2 r_2)\Gamma(-k/2 - z + t + \beta - i\epsilon_1 r_1 - i\epsilon_2 r_2)}{\Gamma(\beta - i\epsilon_1 r_1)} \frac{2tdt}{t^2 - t_2^2}.
\end{aligned}$$

□

Remark 5.9. Consider the expression (5.11). If $\beta - i\epsilon_1 r_1 = 0, -1, -2, \dots$, the poles of $1/\sin(\pi(\beta - i\epsilon_1 r_1))$ are canceled by the zeroes of $1/\Gamma(\beta - i\epsilon_1 r_1)$. The poles at $\beta - i\epsilon_1 r_1 = j$ with $j = 1, 2, 3, \dots$ are compensated by the vanishing numerator.

5.4.2 x-integral

Lemma 5.10. *The function $F(x)$ can be replaced by 1 in the expression $V(h)$ at the cost of negligible error $O_{\epsilon, \mathbf{r}}(q^{\epsilon - k/2 + 0.5})$.*

Proof. We show that the contribution of $F_1(x) = 1 - F(x)$ is negligible. Note that $F_1(x) = 0$ for $x \geq 1$ since in that case $F(x) = 1$. The part of M^{OOD} which affects the x -integral can be written as follows

$$\begin{aligned}
& \sum_{v, w} \frac{1}{v^2 w} \sum_{\substack{c, g \\ q|cg}} \sum_{[c, w]|h} g^{-k-1-2z+2\beta} c^{-k-2z+2\beta} h^{k/2+z-\beta-t} q^t \\
& \times \Gamma(-k/2 - z + t + \beta - i\epsilon_1 r_1 - i\epsilon_2 r_2) \Gamma(k/2 + z - t + i\epsilon_2 r_2) H_1(t, z, \beta) \\
& \times \int_0^1 x^{z-\beta+k/2} W_{t_1, r_1}\left(\frac{x}{\hat{q}^2}\right) F_1(x) \frac{dx}{x}.
\end{aligned}$$

Here H_1 is an analytic function. We have

$$\Re z = -0.1, \Re \beta = 0.7, \Re t = k/2 - 0.2.$$

Without crossing any poles, we shift β -contour to

$$\Re \beta = 0.3.$$

In order to make the sums over h and w absolutely convergent, we move the t contour to

$$\Re t = k/2 + 0.7,$$

crossing a pole at $t = k/2 + z + i\epsilon_2 r_2$. Since $\Re z - \Re \beta + k/2 > 0$, the x -integral can be done by parts n times (for sufficiently large n) to make β -integral convergent. This gives

$$\int_0^1 x^{z-\beta+k/2} W_{t_1, r_1}(\frac{x}{\hat{q}^2}) F_1(x) \frac{dx}{x} \ll \frac{1}{|\beta|^n}.$$

Finally, all sums and integrals are absolutely convergent and $q^{-k-1+t-2z+2\beta}$ can be factored out due to divisibility conditions. In total, this gives an error $O_{\epsilon, \mathbf{r}}(q^{\epsilon-k/2+0.5})$.

For the pole at $t = k/2 + z + i\epsilon_2 r_2$ another contour shift is required to make all sums absolutely convergent. We move the z -contour to

$$\Re z = 0.5 + 2\epsilon$$

and β to

$$\Re \beta = 1 + \epsilon.$$

Note that the pole of $1/\sin(\pi z)$ at $z = 0$ is canceled by zero of $P_r(t) = P_r(k/2 + z + i\epsilon_2 r_2)$. The x integral is bounded by $\frac{1}{|\beta|^n}$. The power of q , corresponding to divisibility conditions on g, c, h , is $q^{-k-1+t-2z+2\beta}$. This gives the error term $O_{\epsilon, \mathbf{r}}(q^{\epsilon-k/2+0.5})$. \square

Proposition 5.11. *One has*

$$\begin{aligned} M^{OOD}(B) &= 2\pi i^k \hat{q}^{-2t_1-2t_2} \sum_{\epsilon_1, \epsilon_2 = \pm 1} \zeta(1 + 2i\epsilon_1 r_1) \zeta(1 + 2i\epsilon_2 r_2) \\ &\times \left(\sum_{g, v} \frac{\mu(g)}{g^2} \frac{\mu(v)}{v^{2+2i\epsilon_1 r_1+2i\epsilon_2 r_2}} \sum_{\substack{q|cg \\ cg < q^\Omega}} \frac{1}{c} \sum_w \frac{(p^B, wv)^{1+2i\epsilon_2 r_2}}{w^{1+2i\epsilon_1 r_1+2i\epsilon_2 r_2}} \sum_{\substack{h \neq 0 \\ [w, c]|h}} V(h) \right. \\ &\quad \left. - \frac{1}{p} \sum_{g, v} \frac{\mu(g)}{g^2} \frac{\mu(v)}{v^{2+2i\epsilon_1 r_1+2i\epsilon_2 r_2}} \sum_{\substack{\frac{q}{p}|cg \\ cg < q^\Omega}} \frac{1}{c} \sum_w \frac{(p^B, wv)^{1+2i\epsilon_2 r_2}}{w^{1+2i\epsilon_1 r_1+2i\epsilon_2 r_2}} \sum_{\substack{h \neq 0 \\ [w, c]|h}} V(h) \right), \end{aligned}$$

where

$$\begin{aligned}
V(h) = & -\frac{i^k}{(2\pi i)^3} \int_{\Re t = k/2 + 0.7} \int_{\Re s = k/2 - 0.4} \int_{\Re z = -0.1} \frac{\hat{q}^{2s+2t}}{p^{B(1+i\epsilon_2 r_2)}} (cg)^{1-2s} \\
& \times (2\pi)^{2s} \frac{P_r(s)P_r(t)}{P_r(t_1)P_r(t_2)} \zeta_q(1+2t)\zeta_q(1+2s) \frac{h^{s-t+i\epsilon_1 r_1+i\epsilon_2 r_2}}{\sin(\pi z)\Gamma(1+z)\Gamma(k+z)} \\
& \times \frac{\Gamma(k/2+s \pm ir_1)\Gamma(k/2+t \pm ir_2)}{\Gamma(k/2+t_1 \pm ir_1)\Gamma(k/2+t_2 \pm ir_2)} \\
& \times \Gamma(t-s-i\epsilon_1 r_1-i\epsilon_2 r_2)\Gamma(k/2+z-s+i\epsilon_1 r_1)\Gamma(k/2+z-t+i\epsilon_2 r_2) \\
& \times \left(\cos(\pi(z-s)) + \frac{\cos(\pi(z-s))\sin(\pi(z-t+i\epsilon_2 r_2))}{\sin(\pi(z-s-i\epsilon_1 r_1))} \right. \\
& \left. + \frac{\cos(\pi ir_1)\sin(\pi(t-s-i\epsilon_1 r_1-i\epsilon_2 r_2))}{\sin(\pi(z-s-i\epsilon_1 r_1))} \right) dz \frac{2sds}{s^2-t_1^2} \frac{2tdt}{t^2-t_2^2}.
\end{aligned}$$

Remark 5.12. We do not compute the contribution of poles at $t = k/2 + z + i\epsilon_2 r_2$ since it will be cancelled by another contour shift in 5.5.1.

Proof. By inverse Mellin transform we have

$$\begin{aligned}
\int_0^\infty x^{z-1-\beta+k/2} W_{t_1, r_1}\left(\frac{x}{\hat{q}^2}\right) dx &= 2 \frac{P_r(z-\beta+k/2)}{P_r(t_1)} \zeta_q(1+k+2z-2\beta) \\
&\times \hat{q}^{k+2z-2\beta} \frac{\Gamma(k+z-\beta+ir_1)\Gamma(k+z-\beta-ir_1)}{\Gamma(k/2+t_1+ir_1)\Gamma(k/2+t_1-ir_1)} \frac{k/2+z-\beta}{(k/2+z-\beta)^2-t_1^2}
\end{aligned}$$

for $\Re(z-\beta+k/2) > -1$. Setting $s := k/2 + z - \beta$ yields the assertion. \square

5.5 Shifting the z -contour

The z -integral is given by

$$\begin{aligned}
(5.12) \quad IZ := & \frac{1}{2\pi i} \int_{\Re z = -0.1} \frac{\Gamma(k/2+z-s+i\epsilon_1 r_1)\Gamma(k/2+z-t+i\epsilon_2 r_2)}{\sin(\pi z)\Gamma(1+z)\Gamma(k+z)} \\
& \times \left(\cos(\pi(z-s)) + \frac{\cos(\pi(z-s))\sin(\pi(z-t+i\epsilon_2 r_2))}{\sin(\pi(z-s-i\epsilon_1 r_1))} \right. \\
& \left. + \frac{\cos(\pi ir_1)\sin(\pi(t-s-i\epsilon_1 r_1-i\epsilon_2 r_2))}{\sin(\pi(z-s-i\epsilon_1 r_1))} \right) dz.
\end{aligned}$$

Stirling's formula implies that the integrand decays as $|z|^{-1-s-t}$. We shift $\Re z$ to $D > 0$ and let $D \rightarrow +\infty$. This leads to three types of possible poles described in the table below.

Possible poles at	Coming from function
$z = t - k/2 - i\epsilon_2 r_2$	$\Gamma(k/2 + z - t + i\epsilon_2 r_2)$
$z = n + s + i\epsilon_1 r_1$	$1/\sin(\pi(z-s-i\epsilon_1 r_1))$
$z = n, n \geq 0$	$1/\sin(\pi z)$

5.5.1 Poles at $z = t - k/2 - i\epsilon_2 r_2$

The residues at these poles cancel those mentioned in remark 5.12 (while performing the shift of t to the right). Consider

$$\int_t \int_z \Gamma(k/2 + z - t + i\epsilon_2 r_2) f(z, t) dz dt.$$

Shifting t integral to the right, we have the residue

$$-Res_{z=-k/2+t-i\epsilon_2 r_2} \Gamma(k/2 + z - t + i\epsilon_2 r_2) f(z, t).$$

Moving z to the right, we obtain

$$-Res_{t=k/2+z+i\epsilon_2 r_2} \Gamma(k/2 + z - t + i\epsilon_2 r_2) f(z, t).$$

Since z and t have different signs in $\Gamma(k/2 + z - t + i\epsilon_2 r_2)$, these residues cancel each other.

5.5.2 Poles at $z = n + s + i\epsilon_1 r_1$

Proposition 5.13. *The expression under the integral in IZ is holomorphic at $z = n + s + i\epsilon_1 r_1$.*

Proof. To show this, we write

$$\begin{aligned} \sin(\pi(z - t + i\epsilon_2 r_2)) &= -\sin(\pi(t - s - i\epsilon_1 r_1 - i\epsilon_2 r_2)) \\ &\times \cos(\pi(z - s - i\epsilon_1 r_1)) + \cos(\pi(t - s - i\epsilon_1 r_1 - i\epsilon_2 r_2)) \sin(\pi(z - s - i\epsilon_1 r_1)) \end{aligned}$$

and plug it in IZ . After simplifications,

$$\begin{aligned} IZ &= \frac{1}{2\pi i} \int_{\Re z = -0.1} \frac{\Gamma(k/2 + z - s + i\epsilon_1 r_1) \Gamma(k/2 + z - t + i\epsilon_2 r_2)}{\sin(\pi z) \Gamma(1 + z) \Gamma(k + z)} \\ &\times \left[\cos(\pi(z - s)) + \cos(\pi(z - s)) \cos(\pi(t - s - i\epsilon_1 r_1 - i\epsilon_2 r_2)) \right. \\ &\quad \left. + \sin(\pi(z - s)) \sin(\pi(z - s + i\epsilon_1 r_1)) \right]. \end{aligned}$$

This is holomorphic at $z = n + s + i\epsilon_1 r_1$. \square

5.5.3 Poles at $z = n, n \geq 0$

Proposition 5.14. *The poles at $z = n$ are simple and its contribution to IZ is given by*

$$\begin{aligned} &-\frac{1}{\pi} \Gamma(s + t - i\epsilon_1 r_1 - i\epsilon_2 r_2) \frac{\Gamma(k/2 - s + i\epsilon_1 r_1) \Gamma(k/2 - t + i\epsilon_2 r_2)}{\Gamma(k/2 + s - i\epsilon_1 r_1) \Gamma(k/2 + t - i\epsilon_2 r_2)} \\ &\quad \times [\cos(\pi s) + \cos(\pi(t - i\epsilon_1 r_1 - i\epsilon_2 r_2))]. \end{aligned}$$

Proof. Consider

$$P_1 := -\frac{1}{\pi} \sum_{n=0}^{\infty} \frac{1}{\cos(\pi n)} \frac{\Gamma(k/2 + n - s + i\epsilon_1 r_1) \Gamma(k/2 + n - t + i\epsilon_2 r_2)}{\Gamma(1+n) \Gamma(k+n)} \\ \times \left(\cos(\pi(n-s)) + \frac{\cos(\pi(n-s)) \sin(\pi(n-t + i\epsilon_2 r_2))}{\sin(\pi(n-s - i\epsilon_1 r_1))} \right. \\ \left. + \frac{\cos(\pi i r_1) \sin(\pi(t-s - i\epsilon_1 r_1 - i\epsilon_2 r_2))}{\sin(\pi(n-s - i\epsilon_1 r_1))} \right).$$

Since $n \in \mathbb{Z}$, we have

$$P_1 := -\frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\Gamma(k/2 + n - s + i\epsilon_1 r_1) \Gamma(k/2 + n - t + i\epsilon_2 r_2)}{\Gamma(1+n) \Gamma(k+n)} \\ \times \left(\cos(\pi s) + \frac{\cos(\pi s) \sin(\pi(t - i\epsilon_2 r_2))}{\sin(\pi(s + i\epsilon_1 r_1))} \right. \\ \left. - \frac{\cos(\pi i r_1) \sin(\pi(t-s - i\epsilon_1 r_1 - i\epsilon_2 r_2))}{\sin(\pi(s + i\epsilon_1 r_1))} \right).$$

By Gauss hypergeometric identity,

$$\sum_{n=0}^{\infty} \frac{\Gamma(k/2 + n - s + i\epsilon_1 r_1) \Gamma(k/2 + n - t + i\epsilon_2 r_2)}{\Gamma(1+n) \Gamma(k+n)} \\ = \Gamma(s+t - i\epsilon_1 r_1 - i\epsilon_2 r_2) \frac{\Gamma(k/2 - s + i\epsilon_1 r_1) \Gamma(k/2 - t + i\epsilon_2 r_2)}{\Gamma(k/2 + s - i\epsilon_1 r_1) \Gamma(k/2 + t - i\epsilon_2 r_2)}.$$

Simplifying the trigonometric part, we obtain

$$\frac{\cos(\pi s) \sin(\pi(t - i\epsilon_2 r_2))}{\sin(\pi(s + i\epsilon_1 r_1))} - \frac{\cos(\pi i r_1) \sin(\pi(t-s - i\epsilon_1 r_1 - i\epsilon_2 r_2))}{\sin(\pi(s + i\epsilon_1 r_1))} \\ = \cos(\pi(t - i\epsilon_1 r_1 - i\epsilon_2 r_2)).$$

This implies

$$P_1 = -\frac{1}{\pi} \Gamma(s+t - i\epsilon_1 r_1 - i\epsilon_2 r_2) \frac{\Gamma(k/2 - s + i\epsilon_1 r_1) \Gamma(k/2 - t + i\epsilon_2 r_2)}{\Gamma(k/2 + s - i\epsilon_1 r_1) \Gamma(k/2 + t - i\epsilon_2 r_2)} \\ \times [\cos(\pi s) + \cos(\pi(t - i\epsilon_1 r_1 - i\epsilon_2 r_2))].$$

□

Proposition 5.15. *The off-off-diagonal term can be written as follows*

$$\begin{aligned}
M^{OOD}(B) &= \frac{2}{(2\pi i)^2} \hat{q}^{-2t_1-2t_2} \sum_{\epsilon_1, \epsilon_2 = \pm 1} \zeta(1+2i\epsilon_1 r_1) \zeta(1+2i\epsilon_2 r_2) \\
&\times \int_{\Re t = k/2+0.7} \int_{\Re s = k/2-0.4} \frac{P_r(s)P_r(t)}{P_r(t_1)P_r(t_2)} \zeta_q(1+2s) \zeta_q(1+2t) \frac{q^{s+t}}{(2\pi)^{2t}} \\
&\times \Gamma(t-s-i\epsilon_1 r_1-i\epsilon_2 r_2) \Gamma(t+s-i\epsilon_1 r_1-i\epsilon_2 r_2) \\
&\times \frac{\Gamma(k/2+s+i\epsilon_1 r_1) \Gamma(k/2+t+i\epsilon_2 r_2) \Gamma(k/2-s+i\epsilon_1 r_1)}{\Gamma(k/2+t_1+ir_1) \Gamma(k/2+t_1-ir_1) \Gamma(k/2+t_2+ir_2)} \\
&\times \frac{\Gamma(k/2-t+i\epsilon_2 r_2)}{\Gamma(k/2+t_2-ir_2)} \left(\sum_{q|cpq} \sum_g \frac{\mu(g)}{g^{2s+1}} TD(c) - 1/p \sum_{q|cpq} \sum_g \frac{\mu(g)}{g^{2s+1}} TD(c) \right) \\
&\times [\cos(\pi s) + \cos(\pi(t-i\epsilon_1 r_1-i\epsilon_2 r_2))] \frac{2sds}{s^2-t_1^2} \frac{2tdt}{t^2-t_2^2},
\end{aligned}$$

where

$$\begin{aligned}
(5.13) \quad TD(c) &= \frac{1}{c^{2s}} \sum_v \frac{\mu(v)}{v^{2+2i\epsilon_1 r_1+2i\epsilon_2 r_2}} \sum_w \frac{(p^B, wv)^{1+2i\epsilon_2 r_2}}{p^{B(1+i\epsilon_2 r_2)} w^{1+2i\epsilon_1 r_1+2i\epsilon_2 r_2}} \\
&\times \sum_{c, w|h} \frac{1}{h^{t-s-i\epsilon_1 r_1-i\epsilon_2 r_2}}.
\end{aligned}$$

5.6 Explicit formula

We start with transforming the off-off-diagonal term.

Proposition 5.16. *One has*

$$\begin{aligned}
(5.14) \quad M^{OOD} &= \sum_{\epsilon_1, \epsilon_2 = \pm 1} \zeta(1+2i\epsilon_1 r_1) \zeta(1+2i\epsilon_2 r_2) \frac{1}{(2\pi i)^2} \\
&\times \int_{\Re t = k/2+0.6} \int_{\Re s = k/2-0.4} E(s, t) \Phi(s, t) 2sds 2tdt,
\end{aligned}$$

where

$$\begin{aligned}
(5.15) \quad E(s, t) &:= \hat{q}^{-2t_1-2t_2} \frac{P_r(s)P_r(t)}{P_r(t_1)P_r(t_2)} \frac{1}{s^2-t_1^2} \frac{1}{t^2-t_2^2} \frac{\Gamma(k/2+s+i\epsilon_1 r_1)}{\Gamma(k/2+t_1+ir_1)} \\
&\times \frac{\Gamma(k/2+t+i\epsilon_2 r_2) \Gamma(k/2-s+i\epsilon_1 r_1) \Gamma(k/2-t+i\epsilon_2 r_2)}{\Gamma(k/2+t_1-ir_1) \Gamma(k/2+t_2+ir_2) \Gamma(k/2+t_2-ir_2)},
\end{aligned}$$

$$\begin{aligned}
(5.16) \quad \Phi(s, t) &:= 2\zeta_q(1+2t) \frac{q^{s+t}}{(2\pi)^{2t}} [\cos(\pi s) + \cos(\pi(t-i\epsilon_1 r_1-i\epsilon_2 r_2))] \\
&\times \Gamma(t-s-i\epsilon_1 r_1-i\epsilon_2 r_2) \Gamma(t+s-i\epsilon_1 r_1-i\epsilon_2 r_2) \sum_{A, B=0}^2 C(A, B) \sum_{q|cp^A} TD(c)
\end{aligned}$$

and coefficients $C(A, B)$ are given in table 1.

Proof. Consider the term $M^{OOD}(B)$. Möbius function does not vanish only if $(q, g) = 1$ or $(q, g) = p$. Then we can write

$$\begin{aligned}
M^{OOD}(B) &= \frac{2}{(2\pi i)^2} \hat{q}^{-2t_1-2t_2} \sum_{\epsilon_1, \epsilon_2 = \pm 1} \zeta(1 + 2i\epsilon_1 r_1) \zeta(1 + 2i\epsilon_2 r_2) \\
&\quad \times \int_{\Re t = k/2 + 0.7} \int_{\Re s = k/2 - 0.4} \frac{P_r(s) P_r(t)}{P_r(t_1) P_r(t_2)} \zeta_q(1 + 2s) \zeta_q(1 + 2t) \frac{q^{s+t}}{(2\pi)^{2t}} \\
&\times \frac{\Gamma(k/2 + s + i\epsilon_1 r_1) \Gamma(k/2 + t + i\epsilon_2 r_2) \Gamma(k/2 - s + i\epsilon_1 r_1) \Gamma(k/2 - t + i\epsilon_2 r_2)}{\Gamma(k/2 + t_1 + ir_1) \Gamma(k/2 + t_1 - ir_1) \Gamma(k/2 + t_2 + ir_2) \Gamma(k/2 + t_2 - ir_2)} \\
&\times \sum_{(q, g)=1} \frac{\mu(g)}{g^{1+2s}} \left[\sum_{q|c} TD(c) - \left(\frac{1}{p^{1+2s}} + \frac{1}{p} \right) \sum_{q|cp} TD(c) + \frac{1}{p^{2+2s}} \sum_{q|cp^2} TD(c) \right] \\
&\quad \times \Gamma(t - s - i\epsilon_1 r_1 - i\epsilon_2 r_2) \Gamma(t + s - i\epsilon_1 r_1 - i\epsilon_2 r_2) \\
&\quad \times [\cos(\pi s) + \cos(\pi(t - i\epsilon_1 r_1 - i\epsilon_2 r_2))] \frac{2s ds}{s^2 - t_1^2} \frac{2t dt}{t^2 - t_2^2}.
\end{aligned}$$

Note that

$$\zeta_q(1 + 2s) \sum_{(q, g)=1} \frac{\mu(g)}{g^{1+2s}} = 1.$$

In order to simplify notations let us denote

$$\begin{aligned}
E(s, t) &:= \hat{q}^{-2t_1-2t_2} \frac{P_r(s) P_r(t)}{P_r(t_1) P_r(t_2)} \frac{1}{s^2 - t_1^2} \frac{1}{t^2 - t_2^2} \frac{\Gamma(k/2 + s + i\epsilon_1 r_1)}{\Gamma(k/2 + t_1 + ir_1)} \\
&\quad \times \frac{\Gamma(k/2 + t + i\epsilon_2 r_2) \Gamma(k/2 - s + i\epsilon_1 r_1) \Gamma(k/2 - t + i\epsilon_2 r_2)}{\Gamma(k/2 + t_1 - ir_1) \Gamma(k/2 + t_2 + ir_2) \Gamma(k/2 + t_2 - ir_2)}.
\end{aligned}$$

This is an even function since G is even. By equation (3.32)

$$M^{OOD} = M^{OOD}(0) - \tau_{1/2+ir_2}(p) M^{OOD}(1) + M^{OOD}(2).$$

Next, we introduce parameter A corresponding to the condition $q|cp^A$. So that

$$M^{OOD} = \sum_{A, B=0}^2 C(A, B) M^{OOD}(A, B),$$

$$\begin{aligned}
M^{OOD}(A, B) &= \frac{2}{(2\pi i)^2} \sum_{\epsilon_1, \epsilon_2 = \pm 1} \zeta(1 + 2i\epsilon_1 r_1) \zeta(1 + 2i\epsilon_2 r_2) \\
&\quad \times \int_{\Re t = k/2 + 0.7} \int_{\Re s = k/2 - 0.4} E(s, t) \zeta_q(1 + 2t) \frac{q^{s+t}}{(2\pi)^{2t}} \\
&\quad \times [\cos(\pi s) + \cos(\pi(t - i\epsilon_1 r_1 - i\epsilon_2 r_2))] \\
&\quad \times \Gamma(t - s - i\epsilon_1 r_1 - i\epsilon_2 r_2) \Gamma(t + s - i\epsilon_1 r_1 - i\epsilon_2 r_2) \sum_{q|cp^A} TD(c) 2s ds 2t dt,
\end{aligned}$$

	$A = 0$	$A = 1$	$A = 2$
$B = 0$	1	$-(1 + p^{2s})p^{-2s-1}$	p^{-2-2s}
$B = 1$	$-\tau_{1/2+ir_2}(p)$	$\tau_{1/2+ir_2}(p)(1 + p^{2s})p^{-2s-1}$	$-\tau_{1/2+ir_2}(p)p^{-2-2s}$
$B = 2$	1	$-(1 + p^{2s})p^{-2s-1}$	p^{-2-2s}

Table 1: Values of coefficients $C(A, B)$

where coefficients $C(A, B)$ are given in the table 1.

□

The next lemma allows removing the divisibility condition $c, w|h$ in the expression $\sum_{q|cp^A} TD(c)$.

Lemma 5.17. *One has*

$$(5.17) \quad \sum_{\substack{c, w \\ p^{\nu-A}|c}} f(c, w) \sum_{c, w|h} g(h) = \sum_{p \nmid u} \mu(u) \\ \times \sum_{\substack{\beta \geq 0 \\ \delta = \max(\nu-A, \beta)}} \sum_{\substack{c \\ p \nmid d \\ p \nmid w}} f(p^{\nu-A}duc, p^\beta duw) \sum_h g(p^\delta u^2 dcwh).$$

Remark 5.18. Recall that $q = p^\nu$, $\nu \geq 3$ and so $\nu - A \geq 1$.

Proof. Consider

$$S := \sum_{\substack{c, w \\ p^{\nu-A}|c}} f(c, w) \sum_{c, w|h} g(h).$$

Let us make the following change of variables

$$c = p^{\nu-A}c_1 = p^{\nu-A}dc_2,$$

$$w = p^\beta w_1 = p^\beta dw_2,$$

$$d = (c_1, w_1) \text{ so that } (c_2, w_2) = 1 \text{ and } p \nmid dw_2,$$

$$h = p^\delta dc_2 w_2 h_1 \text{ where } \delta = \max(\nu - A, \beta).$$

Then

$$S = \sum_{\substack{\beta \geq 0 \\ \delta = \max(\nu-A, \beta)}} \sum_{p \nmid d} \sum_{\substack{c_2 \\ p \nmid w_2 \\ (c_2, w_2)=1}} f(p^{\nu-A}dc_2, p^\beta dw_2) \sum_{h_1} g(p^\delta dc_2 w_2 h_1).$$

Finally, we remove the requirement $(c_2, w_2) = 1$ by Möbius inversion

$$S = \sum_{p \nmid u} \mu(u) \sum_{\substack{\beta \geq 0 \\ \delta = \max(\nu-A, \beta)}} \sum_{\substack{c \\ p \nmid d \\ p \nmid w}} f(p^{\nu-A}duc, p^\beta duw) \sum_h g(p^\delta u^2 dcwh).$$

□

Proposition 5.19. *We have*

$$\begin{aligned} \Phi(s, t) &= q^{t-s} (2\pi)^{-2i\epsilon_1 r_1 - 2i\epsilon_2 r_2} \frac{\zeta_q(1+t+s+i\epsilon_1 r_1+i\epsilon_2 r_2)}{\zeta_q(2+2i\epsilon_1 r_1+2i\epsilon_2 r_2)} \\ &\quad \times \zeta(1-t+s+i\epsilon_1 r_1+i\epsilon_2 r_2) \zeta(1-t-s+i\epsilon_1 r_1+i\epsilon_2 r_2) \\ &\quad \times \zeta_q(1+t-s+i\epsilon_1 r_1+i\epsilon_2 r_2) \sum_{\alpha \geq 0} \frac{\mu(p^\alpha)}{p^{\alpha(2+2i\epsilon_1 r_1+2i\epsilon_2 r_2)}} \sum_{A, B=0}^2 C(A, B) (p^A)^{2s} \\ &\quad \times \sum_{\substack{\beta \geq 0 \\ \delta = \max(\nu-A, \beta)}} \frac{(p^B, p^{\alpha+\beta})^{1+2i\epsilon_2 r_2}}{p^{B(1+i\epsilon_2 r_2)} p^{\beta(1+2i\epsilon_1 r_1+2i\epsilon_2 r_2)} p^{\delta(t-s-i\epsilon_1 r_1-i\epsilon_2 r_2)}}. \end{aligned}$$

Proof. The expression $TD(c)$ is given by (5.13). Consider

$$\begin{aligned} \sum_{p^{\nu-A}|c} TD(c) &= \sum_{p^{\nu-A}|c} \frac{1}{c^{2s}} \sum_v \frac{\mu(v)}{v^{2+2i\epsilon_1 r_1+2i\epsilon_2 r_2}} \sum_w \frac{(p^B, wv)^{1+2i\epsilon_2 r_2}}{p^{B(1+i\epsilon_2 r_2)} w^{1+2i\epsilon_1 r_1+2i\epsilon_2 r_2}} \\ &\quad \times \sum_{c, w|h} \frac{1}{h^{t-s-i\epsilon_1 r_1-i\epsilon_2 r_2}}. \end{aligned}$$

According to the lemma 5.17

$$c \rightarrow p^{\nu-A} duc,$$

$$w \rightarrow p^\beta duw,$$

$$h \rightarrow p^\delta u^2 dcwh.$$

In addition, the sum over v can be decomposed as

$$\sum_v \frac{\mu(v)}{v^{2+2i\epsilon_1 r_1+2i\epsilon_2 r_2}} = \sum_{\alpha \geq 0} \frac{\mu(p^\alpha)}{p^{\alpha(2+2i\epsilon_1 r_1+2i\epsilon_2 r_2)}} \sum_{(v,p)=1} \frac{\mu(v)}{v^{2+2i\epsilon_1 r_1+2i\epsilon_2 r_2}}.$$

Then

$$\begin{aligned} \sum_{q|cp^A} TD(c) &= \left(\frac{p^A}{q}\right)^{2s} \sum_{(u,p)=1} \frac{\mu(u)}{u^{2t+1}} \sum_{(v,p)=1} \frac{\mu(v)}{v^{2+2i\epsilon_1 r_1+2i\epsilon_2 r_2}} \\ &\quad \times \sum_c \frac{1}{c^{t+s-i\epsilon_1 r_1-i\epsilon_2 r_2}} \sum_h \frac{1}{h^{t-s-i\epsilon_1 r_1-i\epsilon_2 r_2}} \sum_{(d,p)=1} \frac{1}{d^{1+t+s+i\epsilon_1 r_1+i\epsilon_2 r_2}} \\ &\quad \times \sum_{(w,p)=1} \frac{1}{w^{1+t-s+i\epsilon_1 r_1+i\epsilon_2 r_2}} \sum_{\alpha \geq 0} \frac{\mu(p^\alpha)}{p^{\alpha(2+2i\epsilon_1 r_1+2i\epsilon_2 r_2)}} \\ &\quad \times \sum_{\substack{\beta \geq 0 \\ \delta = \max(\nu-A, \beta)}} \frac{(p^B, p^{\alpha+\beta})^{1+2i\epsilon_2 r_2}}{p^{B(1+i\epsilon_2 r_2)} p^{\beta(1+2i\epsilon_1 r_1+2i\epsilon_2 r_2)} p^{\delta(t-s-i\epsilon_1 r_1-i\epsilon_2 r_2)}}. \end{aligned}$$

The asymmetric functional equation implies

$$\begin{aligned} & \frac{\Gamma(t-s-i\epsilon_1 r_1-i\epsilon_2 r_2)\Gamma(t+s-i\epsilon_1 r_1-i\epsilon_2 r_2)\prod\zeta(t\pm s-i\epsilon_1 r_1-i\epsilon_2 r_2)}{(2\pi)^{2t-2i\epsilon_1 r_1-2i\epsilon_2 r_2}} \\ &= \frac{\zeta(1-t-s+i\epsilon_1 r_1+i\epsilon_2 r_2)\zeta(1-t+s+i\epsilon_1 r_1+i\epsilon_2 r_2)}{2[\cos(\pi s)+\cos(\pi(t-i\epsilon_1 r_1-i\epsilon_2 r_2))]} \end{aligned}$$

Thus,

$$\begin{aligned} \Phi(s, t) &= q^{t-s}(2\pi)^{-2i\epsilon_1 r_1-2i\epsilon_2 r_2} \frac{\zeta_q(1+t+s+i\epsilon_1 r_1+i\epsilon_2 r_2)}{\zeta_q(2+2i\epsilon_1 r_1+2i\epsilon_2 r_2)} \\ &\times \zeta_q(1+t-s+i\epsilon_1 r_1+i\epsilon_2 r_2)\zeta(1-t+s+i\epsilon_1 r_1+i\epsilon_2 r_2)\zeta(1-t-s+i\epsilon_1 r_1+i\epsilon_2 r_2) \\ &\times \sum_{\alpha \geq 0} \frac{\mu(p^\alpha)}{p^{\alpha(2+2i\epsilon_1 r_1+2i\epsilon_2 r_2)}} \sum_{A, B=0}^2 C(A, B)(p^A)^{2s} \\ &\times \sum_{\substack{\beta \geq 0 \\ \delta = \max(\nu-A, \beta)}} \frac{(p^B, p^{\alpha+\beta})^{1+2i\epsilon_2 r_2}}{p^{B(1+i\epsilon_2 r_2)} p^{\beta(1+2i\epsilon_1 r_1+2i\epsilon_2 r_2)} p^{\delta(t-s-i\epsilon_1 r_1-i\epsilon_2 r_2)}}. \end{aligned}$$

□

The sums over α and β in $\Phi(s, t)$ can be evaluated by considering different cases, as we now show.

5.6.1 Case 1: $\beta > \nu - A$

Proposition 5.20. *The given case contributes to the off-off-diagonal term as $O_{\epsilon, r}(q^{-1+\epsilon})$.*

Proof. We have $\delta = \beta$ and

$$\begin{aligned} \Phi(s, t) &= q^{t-s}(2\pi)^{-2i\epsilon_1 r_1-2i\epsilon_2 r_2} \frac{\zeta_q(1+t+s+i\epsilon_1 r_1+i\epsilon_2 r_2)}{\zeta_q(2+2i\epsilon_1 r_1+2i\epsilon_2 r_2)} \\ &\times \zeta(1-t+s+i\epsilon_1 r_1+i\epsilon_2 r_2)\zeta(1-t-s+i\epsilon_1 r_1+i\epsilon_2 r_2) \\ &\times \zeta_q(1+t-s+i\epsilon_1 r_1+i\epsilon_2 r_2) \sum_{\alpha \geq 0} \frac{\mu(p^\alpha)}{p^{\alpha(2+2i\epsilon_1 r_1+2i\epsilon_2 r_2)}} \sum_{A, B=0}^2 C(A, B)(p^A)^{2s} \\ &\times \sum_{\beta \geq \nu-A+1} \frac{(p^B, p^{\alpha+\beta})^{1+2i\epsilon_2 r_2}}{p^{B(1+i\epsilon_2 r_2)} p^{\beta(1+t-s+i\epsilon_1 r_1+i\epsilon_2 r_2)}}. \end{aligned}$$

The sum over β is given by

$$\begin{aligned} q^{t-s} \sum_{\beta \geq \nu-A+1} \frac{1}{(p^{1+t-s+i\epsilon_1 r_1+i\epsilon_2 r_2})^\beta} &= \frac{1}{q} (p^{A-1})^{1+t-s+i\epsilon_1 r_1+i\epsilon_2 r_2} \\ &\times \sum_{\beta \geq 0} \frac{1}{(p^{1+t-s+i\epsilon_1 r_1+i\epsilon_2 r_2})^\beta}. \end{aligned}$$

This implies that the contribution of this case to M^{OOD} is bounded by $O_{\epsilon, r}(q^{-1+\epsilon})$. \square

5.6.2 Case 2: $\beta \leq \nu - A$

The condition $\beta \leq \nu - A$ means that $\delta = \nu - A$ and

$$\begin{aligned} \Phi(s, t) &= \hat{q}^{2i\epsilon_1 r_1 + 2i\epsilon_2 r_2} \frac{\zeta_q(1 + t + s + i\epsilon_1 r_1 + i\epsilon_2 r_2)}{\zeta_q(2 + 2i\epsilon_1 r_1 + 2i\epsilon_2 r_2)} \\ &\quad \times \zeta(1 - t + s + i\epsilon_1 r_1 + i\epsilon_2 r_2) \zeta(1 - t - s + i\epsilon_1 r_1 + i\epsilon_2 r_2) \\ &\quad \times \zeta_q(1 + t - s + i\epsilon_1 r_1 + i\epsilon_2 r_2) \sum_{\alpha \geq 0} \frac{\mu(p^\alpha)}{p^{\alpha(2 + 2i\epsilon_1 r_1 + 2i\epsilon_2 r_2)}} \\ &\quad \times \sum_{A, B=0}^2 C(A, B) (p^A)^{t+s-i\epsilon_1 r_1-i\epsilon_2 r_2} \sum_{0 \leq \beta \leq \nu-A} \frac{(p^B, p^{\alpha+\beta})^{1+2i\epsilon_2 r_2}}{p^{B(1+i\epsilon_2 r_2)} p^{\beta(1+2i\epsilon_1 r_1+2i\epsilon_2 r_2)}}. \end{aligned}$$

The sum over β can be decomposed in the following way:

$$\begin{aligned} \sum_{0 \leq \beta \leq \nu-A} \frac{(p^B, p^{\alpha+\beta})^{1+2i\epsilon_2 r_2}}{p^{B(1+i\epsilon_2 r_2)} p^{\beta(1+2i\epsilon_1 r_1+2i\epsilon_2 r_2)}} &= \sum_{\substack{0 \leq \beta \leq \nu-A \\ B \leq \alpha+\beta}} \frac{(p^B)^{i\epsilon_2 r_2}}{p^{\beta(1+2i\epsilon_1 r_1+2i\epsilon_2 r_2)}} \\ &+ \sum_{\substack{0 \leq \beta \leq \nu-A \\ B > \alpha+\beta}} \frac{(p^\alpha)^{1+2i\epsilon_2 r_2}}{p^{B(1+i\epsilon_2 r_2)} p^{\beta(2i\epsilon_1 r_1)}} = \sum_{0 \leq \beta \leq \nu-A} \frac{(p^B)^{i\epsilon_2 r_2}}{p^{\beta(1+2i\epsilon_1 r_1+2i\epsilon_2 r_2)}} \\ &- \sum_{\substack{0 \leq \beta \leq \nu-A \\ B > \alpha+\beta}} \frac{(p^B)^{i\epsilon_2 r_2}}{p^{\beta(1+2i\epsilon_1 r_1+2i\epsilon_2 r_2)}} + \sum_{\substack{0 \leq \beta \leq \nu-A \\ B > \alpha+\beta}} \frac{(p^\alpha)^{1+2i\epsilon_2 r_2}}{p^{B(1+i\epsilon_2 r_2)} p^{\beta(2i\epsilon_1 r_1)}}. \end{aligned}$$

The first sum does not contribute to $\Phi(s, t)$ because

$$\sum_{0 \leq \beta \leq \nu-A} \frac{1}{p^{\beta(1+2i\epsilon_1 r_1+2i\epsilon_2 r_2)}} = \left(1 - \frac{1}{p^{1+2i\epsilon_1 r_1+2i\epsilon_2 r_2}}\right)^{-1} \left(1 + O\left(\frac{1}{q}\right)\right)$$

and

$$\sum_{A, B=0}^2 C(A, B) (p^A)^{t+s-i\epsilon_1 r_1-i\epsilon_2 r_2} (p^B)^{i\epsilon_2 r_2} = 0.$$

Therefore,

$$\begin{aligned} \Phi(s, t) &= \hat{q}^{2i\epsilon_1 r_1 + 2i\epsilon_2 r_2} \frac{\zeta_q(1+t+s+i\epsilon_1 r_1+i\epsilon_2 r_2)}{\zeta_q(2+2i\epsilon_1 r_1+2i\epsilon_2 r_2)} \\ &\quad \times \zeta(1-t+s+i\epsilon_1 r_1+i\epsilon_2 r_2) \zeta(1-t-s+i\epsilon_1 r_1+i\epsilon_2 r_2) \\ &\quad \times \zeta_q(1+t-s+i\epsilon_1 r_1+i\epsilon_2 r_2) \sum_{A, B=0}^2 C(A, B) (p^A)^{t+s-i\epsilon_1 r_1-i\epsilon_2 r_2} \\ &\quad \times \sum_{\alpha \geq 0} \frac{\mu(p^\alpha)}{p^{\alpha(2+2i\epsilon_1 r_1+2i\epsilon_2 r_2)}} \sum_{\substack{0 \leq \beta \leq \nu-A \\ B > \alpha + \beta}} \left(\frac{-(p^B)^{i\epsilon_2 r_2}}{p^{\beta(1+2i\epsilon_1 r_1+2i\epsilon_2 r_2)}} + \frac{(p^\alpha)^{1+2i\epsilon_2 r_2}}{p^{B(1+i\epsilon_2 r_2)} p^{\beta(2i\epsilon_1 r_1)}} \right). \end{aligned}$$

For each fixed B the sum over A can be evaluated using table 1:

$$\sum_{A=0}^2 C(A, 0) (p^A)^{t+s-i\epsilon_1 r_1-i\epsilon_2 r_2} = (1 - p^{t+s+1-i\epsilon_1 r_1-i\epsilon_2 r_2}) (1 - p^{t-s+1-i\epsilon_1 r_1-i\epsilon_2 r_2}),$$

$$\begin{aligned} \sum_{A=0}^2 C(A, 1) (p^A)^{t+s-i\epsilon_1 r_1-i\epsilon_2 r_2} &= -(p^{ir_2} + p^{-ir_2}) (1 - p^{t+s+1-i\epsilon_1 r_1-i\epsilon_2 r_2}) \\ &\quad \times (1 - p^{t-s+1-i\epsilon_1 r_1-i\epsilon_2 r_2}), \end{aligned}$$

$$\sum_{A=0}^2 C(A, 2) (p^A)^{t+s-i\epsilon_1 r_1-i\epsilon_2 r_2} = (1 - p^{t+s+1-i\epsilon_1 r_1-i\epsilon_2 r_2}) (1 - p^{t-s+1-i\epsilon_1 r_1-i\epsilon_2 r_2}).$$

Since $B = 0, 1, 2$, the requirement $B > \alpha + \beta$ is satisfied in four cases

$$(B, \alpha, \beta) = \{(1, 0, 0), (2, 0, 0), (2, 1, 0), (2, 0, 1)\}.$$

Thus,

$$\begin{aligned} \Phi(s, t) &= \hat{q}^{2i\epsilon_1 r_1 + 2i\epsilon_2 r_2} \\ &\quad \times \frac{\zeta_q(1+t+s+i\epsilon_1 r_1+i\epsilon_2 r_2) \zeta_q(1+t-s+i\epsilon_1 r_1+i\epsilon_2 r_2)}{\zeta_q(2+2i\epsilon_1 r_1+2i\epsilon_2 r_2)} \\ &\quad \times \zeta_q(1-t+s+i\epsilon_1 r_1+i\epsilon_2 r_2) \zeta_q(1-t-s+i\epsilon_1 r_1+i\epsilon_2 r_2) \\ &\quad \times \left[(p^{ir_2} + \frac{1}{p^{ir_2}}) (p^{i\epsilon_2 r_2} - \frac{1}{p^{1+i\epsilon_2 r_2}}) - p^{2i\epsilon_2 r_2} + \frac{1}{p^{2+2i\epsilon_2 r_2}} \right. \\ &\quad \left. + \frac{1}{p^{2+2i\epsilon_1 r_1}} - \frac{1}{p^{3+2i\epsilon_1 r_1+2i\epsilon_2 r_2}} - \frac{1}{p^{1+2i\epsilon_1 r_1}} + \frac{1}{p^{2+2i\epsilon_1 r_1+2i\epsilon_2 r_2}} \right]. \end{aligned}$$

Simplifying, we have

$$\begin{aligned} \Phi(s, t) &= \frac{\phi(q)}{q} \hat{q}^{2i\epsilon_1 r_1 + 2i\epsilon_2 r_2} \left(1 - \frac{1}{p^{1+2i\epsilon_1 r_1}} \right) \left(1 - \frac{1}{p^{1+2i\epsilon_2 r_2}} \right) \\ &\quad \times \frac{\zeta_q(1+t+s+i\epsilon_1 r_1+i\epsilon_2 r_2) \zeta_q(1+t-s+i\epsilon_1 r_1+i\epsilon_2 r_2)}{\zeta_q(2+2i\epsilon_1 r_1+2i\epsilon_2 r_2)} \\ &\quad \times \zeta_q(1-t+s+i\epsilon_1 r_1+i\epsilon_2 r_2) \zeta_q(1-t-s+i\epsilon_1 r_1+i\epsilon_2 r_2). \end{aligned}$$

Substituting this result in (5.15), we prove theorem 5.1.

6 Off-off-diagonal term: asymptotic evaluation

Theorem 6.1. *Up to a negligible error term, we have*

$$(6.1) \quad M^{OOD} = \frac{\phi(q)}{q} \sum_{\epsilon_1, \epsilon_2 = \pm 1} \frac{\zeta_q(1 + 2i\epsilon_1 r_1) \zeta_q(1 + 2i\epsilon_2 r_2)}{\zeta_q(2 + 2i\epsilon_1 r_1 + 2i\epsilon_2 r_2)} \\ \times \prod_{\epsilon_3, \epsilon_4 = \pm 1} \zeta_q(1 + \epsilon_3 t_1 + \epsilon_4 t_2 + i\epsilon_1 r_1 + i\epsilon_2 r_2) \hat{q}^{-2t_1 - 2t_2 + 2i\epsilon_1 r_1 + 2i\epsilon_2 r_2} \\ \times \frac{\Gamma(k/2 - t_1 + i\epsilon_1 r_1) \Gamma(k/2 - t_2 + i\epsilon_2 r_2)}{\Gamma(k/2 + t_1 - i\epsilon_1 r_1) \Gamma(k/2 + t_2 - i\epsilon_2 r_2)}.$$

Proof. Consider

$$M^{OOD} = \frac{\phi(q)}{q} \sum_{\epsilon_1, \epsilon_2 = \pm 1} \frac{\zeta_q(1 + 2i\epsilon_1 r_1) \zeta_q(1 + 2i\epsilon_2 r_2)}{\zeta_q(2 + 2i\epsilon_1 r_1 + 2i\epsilon_2 r_2)} \hat{q}^{-2t_1 - 2t_2 + 2i\epsilon_1 r_1 + 2i\epsilon_2 r_2} \\ \times \frac{1}{(2\pi i)^2} \int_{\Re t = k/2 + 0.7} \int_{\Re s = k/2 - 0.4} I_{\epsilon_1, \epsilon_2}(s, t) \frac{2s ds}{s^2 - t_1^2} \frac{2t dt}{t^2 - t_2^2},$$

where

$$I_{\epsilon_1, \epsilon_2}(s, t) = \frac{P_r(s) P_r(t)}{P_r(t_1) P_r(t_2)} \zeta_q(1 + t + s + i\epsilon_1 r_1 + i\epsilon_2 r_2) \zeta_q(1 + t - s + i\epsilon_1 r_1 + i\epsilon_2 r_2) \\ \times \zeta_q(1 - t + s + i\epsilon_1 r_1 + i\epsilon_2 r_2) \zeta_q(1 - t - s + i\epsilon_1 r_1 + i\epsilon_2 r_2) \\ \times \frac{\Gamma(k/2 + s + i\epsilon_1 r_1) \Gamma(k/2 + t + i\epsilon_2 r_2) \Gamma(k/2 - s + i\epsilon_1 r_1) \Gamma(k/2 - t + i\epsilon_2 r_2)}{\Gamma(k/2 + t_1 + i r_1) \Gamma(k/2 + t_1 - i r_1) \Gamma(k/2 + t_2 + i r_2) \Gamma(k/2 + t_2 - i r_2)}.$$

The function $I_{\epsilon_1, \epsilon_2}(s, t)$ is even in both s and t . Therefore,

$$4 \frac{1}{(2\pi i)^2} \int_{\Re t = k/2 + 0.7} \int_{\Re s = k/2 - 0.4} I_{\epsilon_1, \epsilon_2}(s, t) \frac{2s ds}{s^2 - t_1^2} \frac{2t dt}{t^2 - t_2^2} \\ = \text{Res}_{\substack{s=t_1 \\ t=t_2}} I(s, t) \frac{2s}{s^2 - t_1^2} \frac{2t}{t^2 - t_2^2} + \text{Res}_{\substack{s=t_1 \\ t=-t_2}} I(s, t) \frac{2s}{s^2 - t_1^2} \frac{2t}{t^2 - t_2^2} \\ + \text{Res}_{\substack{s=-t_1 \\ t=t_2}} I(s, t) \frac{2s}{s^2 - t_1^2} \frac{2t}{t^2 - t_2^2} + \text{Res}_{\substack{s=-t_1 \\ t=-t_2}} I(s, t) \frac{2s}{s^2 - t_1^2} \frac{2t}{t^2 - t_2^2}.$$

Each of the four given residues has the same value. Computing the residue yields the assertion of our theorem. \square

Theorem 6.2. *Up to a negligible error, we have*

$$(6.2) \quad \lim_{(\mathbf{t}, \mathbf{r}) \rightarrow (0,0)} M^{OOD} = \frac{1}{(2\pi i)^2} \int_{\Re t = k/2 + 0.7} \int_{\Re s = k/2 - 0.4} g(s, t) \frac{2ds}{s} \frac{2dt}{t},$$

where

$$\begin{aligned}
g(s, t) = & \left(\frac{\phi(q)}{q} \right)^3 \frac{1}{\zeta_q(2)} \frac{P_r(s)P_r(t)}{P_r(0)^2} \prod_{\epsilon_1, \epsilon_2 = \pm 1} \zeta_q(1 + \epsilon_1 t + \epsilon_2 s) \\
& \times \frac{\Gamma(k/2 + \epsilon_1 t) \Gamma(k/2 + \epsilon_2 s)}{\Gamma(k/2)^4} \left[(2 \log \hat{q} + \gamma)^2 + \sum_{\epsilon_1, \epsilon_2 = \pm 1} \frac{\zeta_q''}{\zeta_q} (1 + \epsilon_1 t + \epsilon_2 s) \right. \\
& + 2 \sum_{\substack{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 = \pm 1 \\ (\epsilon_1, \epsilon_2) \neq (\epsilon_3, \epsilon_4)}} \frac{\zeta_q'}{\zeta_q} (1 + \epsilon_1 t + \epsilon_2 s) \frac{\zeta_q'}{\zeta_q} (1 + \epsilon_3 t + \epsilon_4 s) + (2 \log \hat{q} + \gamma) \\
& \times \left(4 \frac{\zeta_q'}{\zeta_q} (2) - 2 \sum_{\epsilon_1, \epsilon_2 = \pm 1} \frac{\zeta_q'}{\zeta_q} (1 + \epsilon_1 t + \epsilon_2 s) - \sum_{\epsilon = \pm 1} \frac{\Gamma'}{\Gamma} (k/2 + \epsilon s) \right. \\
& \quad \left. - \sum_{\epsilon = \pm 1} \frac{\Gamma'}{\Gamma} (k/2 + \epsilon t) \right) + \sum_{\epsilon_1, \epsilon_2 = \pm 1} \frac{\zeta_q'}{\zeta_q} (1 + \epsilon_1 t + \epsilon_2 s) \\
& \times \left(-4 \frac{\zeta_q'}{\zeta_q} (2) + \sum_{\epsilon = \pm 1} \frac{\Gamma'}{\Gamma} (k/2 + \epsilon t) + \sum_{\epsilon = \pm 1} \frac{\Gamma'}{\Gamma} (k/2 + \epsilon s) \right) \\
& + \sum_{\epsilon = \pm 1} \frac{\Gamma'}{\Gamma} (k/2 + \epsilon t) \sum_{\epsilon = \pm 1} \frac{\Gamma'}{\Gamma} (k/2 + \epsilon s) - 4 \frac{\zeta_q''}{\zeta_q} (2) + 8 \left(\frac{\zeta_q'}{\zeta_q} (2) \right)^2 \\
& \quad \left. - 2 \frac{\zeta_q'}{\zeta_q} (2) \left(\sum_{\epsilon = \pm 1} \frac{\Gamma'}{\Gamma} (k/2 + \epsilon t) \sum_{\epsilon = \pm 1} \frac{\Gamma'}{\Gamma} (k/2 + \epsilon s) \right) \right].
\end{aligned}$$

Corollary 6.3. *The off-off-diagonal term at the critical point is a polynomial in $\log q$ of order 2.*

Proof. First, we let $t_1, t_2 \rightarrow 0$. Then

$$\begin{aligned}
\lim_{t \rightarrow 0} M^{OOD} = & \frac{\phi(q)}{q} \sum_{\epsilon_1, \epsilon_2 = \pm 1} \frac{\zeta_q(1 + 2i\epsilon_1 r_1) \zeta_q(1 + 2i\epsilon_2 r_2)}{\zeta_q(2 + 2i\epsilon_1 r_1 + 2i\epsilon_2 r_2)} \hat{q}^{2i\epsilon_1 r_1 + 2i\epsilon_2 r_2} \\
& \times \frac{1}{(2\pi i)^2} \int_{\Re t = k/2 + 0.7} \int_{\Re s = k/2 - 0.4} I_{\epsilon_1, \epsilon_2}(s, t) \frac{2ds}{s} \frac{2dt}{t},
\end{aligned}$$

where

$$\begin{aligned}
I_{\epsilon_1, \epsilon_2}(s, t) = & \frac{P_r(s)P_r(t)}{P_r(0)^2} \zeta_q(1 + t + s + i\epsilon_1 r_1 + i\epsilon_2 r_2) \\
& \times \zeta_q(1 + t - s + i\epsilon_1 r_1 + i\epsilon_2 r_2) \zeta_q(1 - t + s + i\epsilon_1 r_1 + i\epsilon_2 r_2) \zeta_q(1 - t - s + i\epsilon_1 r_1 + i\epsilon_2 r_2) \\
& \times \frac{\Gamma(k/2 + s + i\epsilon_1 r_1) \Gamma(k/2 + t + i\epsilon_2 r_2) \Gamma(k/2 - s + i\epsilon_1 r_1) \Gamma(k/2 - t + i\epsilon_2 r_2)}{\Gamma(k/2 + ir_1) \Gamma(k/2 - ir_1) \Gamma(k/2 + ir_2) \Gamma(k/2 - ir_2)}.
\end{aligned}$$

Let

$$f(r_1, r_2) := \frac{\phi(q)}{q} \frac{P_r(s)P_r(t)}{P_r(0)^2} \frac{\zeta_q(1+t+s+ir_1+ir_2)}{\zeta_q(2+2ir_1+2ir_2)} \\ \times \zeta_q(1+t-s+ir_1+ir_2)\zeta_q(1-t+s+ir_1+ir_2)\zeta_q(1-t-s+ir_1+ir_2) \\ \times \frac{\Gamma(k/2+s+ir_1)\Gamma(k/2+t+ir_2)\Gamma(k/2-s+ir_1)\Gamma(k/2-t+ir_2)}{\Gamma(k/2+ir_1)\Gamma(k/2-ir_1)\Gamma(k/2+ir_2)\Gamma(k/2-ir_2)}.$$

Consider

$$g(s, t) = \lim_{r_1 \rightarrow 0} \lim_{r_2 \rightarrow 0} \sum_{\epsilon_1, \epsilon_2 = \pm 1} \zeta_q(1+2i\epsilon_1 r_1) \zeta_q(1+2i\epsilon_2 r_2) \hat{q}^{2i\epsilon_1 r_1 + 2i\epsilon_2 r_2} \\ \times f(\epsilon_1 r_1, \epsilon_2 r_2) = \left(\frac{\phi(q)}{q} \right)^2 \left[(2 \log \hat{q} + \gamma)^2 f(0, 0) \right. \\ \left. + i(2 \log \hat{q} + \gamma) \left(\frac{\partial f}{\partial r_1}(0, 0) + \frac{\partial f}{\partial r_2}(0, 0) \right) - \frac{\partial^2 f}{\partial r_1 \partial r_2}(0, 0) \right].$$

Here

$$\frac{\partial f}{\partial r_1}(0, 0) = -if(0, 0) \left(2 \frac{\zeta'_q}{\zeta_q}(2) - \sum \frac{\zeta'_q}{\zeta_q}(1 \pm t \pm s) - \sum \frac{\Gamma'}{\Gamma}(k/2 \pm s) \right), \\ \frac{\partial f}{\partial r_2}(0, 0) = -if(0, 0) \left(2 \frac{\zeta'_q}{\zeta_q}(2) - \sum \frac{\zeta'_q}{\zeta_q}(1 \pm t \pm s) - \sum \frac{\Gamma'}{\Gamma}(k/2 \pm t) \right), \\ \frac{\partial^2 f}{\partial r_1 \partial r_2}(0, 0) = -f(0, 0) \left[\sum \frac{\zeta''_q}{\zeta_q}(1 \pm t \pm s) \right. \\ + 2 \sum^* \frac{\zeta'_q}{\zeta_q}(1 \pm t \pm s) \frac{\zeta'_q}{\zeta_q}(1 \pm t \pm s) + \sum \frac{\zeta'_q}{\zeta_q}(1 \pm t \pm s) \\ \times \left(-4 \frac{\zeta'_q}{\zeta_q}(2) + \sum \frac{\Gamma'}{\Gamma}(k/2 \pm t) + \sum \frac{\Gamma'}{\Gamma}(k/2 \pm s) \right) \\ + \sum \frac{\Gamma'}{\Gamma}(k/2 \pm t) \sum \frac{\Gamma'}{\Gamma}(k/2 \pm s) - 4 \frac{\zeta''_q}{\zeta_q}(2) + 8 \left(\frac{\zeta'_q}{\zeta_q}(2) \right)^2 \\ \left. - 2 \frac{\zeta'_q}{\zeta_q}(2) \left(\sum \frac{\Gamma'}{\Gamma}(k/2 \pm t) \sum \frac{\Gamma'}{\Gamma}(k/2 \pm s) \right) \right].$$

Then

$$\lim_{(\mathbf{t}, \mathbf{r}) \rightarrow (0, 0)} M^{OOD} = \frac{1}{(2\pi i)^2} \int_{\Re t = k/2 + 0.7} \int_{\Re s = k/2 - 0.4} g(s, t) \frac{2ds}{s} \frac{2dt}{t}.$$

The function $g(s, t)$ is even in both variables s and t . Therefore,

$$\lim_{(\mathbf{t}, \mathbf{r}) \rightarrow (0, 0)} M^{OOD} = \frac{1}{4} \operatorname{Res}_{s=t=0} \frac{4g(s, t)}{st} = \operatorname{Res}_{t=0} \frac{g(0, t)}{t}.$$

To find the order of the leading term, we replace all $\zeta(1 \pm t)$ by $\frac{1}{\pm t}$. Let

$$r(t) := \frac{P_r(t)}{P_r(0)} \frac{\Gamma(k/2 + t)\Gamma(k/2 - t)}{\Gamma(k/2)^2}.$$

Then

$$\begin{aligned} \left(\frac{\phi(q)}{q}\right)^7 \frac{1}{\zeta_q(2)} \operatorname{Res}_{t=0} \frac{r(t)}{t^5} \left((\log q)^2 + \frac{4}{t^2}\right) \\ = \left(\frac{\phi(q)}{q}\right)^7 \frac{1}{\zeta_q(2)} \frac{1}{6!} (4r^{(6)}(0) + 30r^{(4)}(0)(\log q)^2). \end{aligned}$$

Therefore, $\lim_{(t,r) \rightarrow (0,0)} M^{OOD}$ is a polynomial in $\log q$ of order 2.

□

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Appendices

A Bessel functions

Lemma A.1. (*[KMV2], Lemma C.1*) Let $z > 0$ and $v \in \mathbb{C}$. Then

$$(A.1) \quad (z^v J_v(z))' = z^v J_{v-1}(z),$$

$$(A.2) \quad (z^v Y_v(z))' = z^v Y_{v-1}(z),$$

$$(A.3) \quad (z^v K_v(z))' = -z^v K_{v-1}(z).$$

Lemma A.2. ([KMV2], Lemma C.2) For $z > 0$ and $j \geq 0$ we have

$$(A.4) \quad \frac{z^j}{(1+z)^j} J_v^{(j)}(z) \ll_{j,v} \frac{z^{\Re v}}{(1+z)^{\Re v+1/2}},$$

$$(A.5) \quad \frac{z^j}{(1+z)^j} Y_0^{(j)}(z) \ll_j \frac{(1+|\log z|)}{(1+z)^{1/2}},$$

$$(A.6) \quad \frac{z^j}{(1+z)^j} K_v^{(j)}(z) \ll_{j,v} \frac{e^{-z}(1+|\log z|)}{(1+z)^{1/2}} \text{ if } \Re v = 0.$$

Lemma A.3. ([Wa], page 149) Assume that $\Re(\mu_1 + \mu_2 + 1) > \Re(2s) > 0$. Then

$$(A.7) \quad \int_0^\infty \frac{J_{\mu_1}(z) J_{\mu_2}(z)}{z^{2s}} dz = \frac{1}{2^{2s}} \frac{\Gamma(2s)}{\Gamma(-\mu_1/2 + \mu_2/2 + s + 1/2)} \\ \times \frac{\Gamma(\mu_1/2 + \mu_2/2 - s + 1/2)}{\Gamma(\mu_1/2 + \mu_2/2 + s + 1/2) \Gamma(\mu_1/2 - \mu_2/2 + s + 1/2)}.$$

Lemma A.4. ([BH], lemma 3) Let $F : (0, \infty) \rightarrow \mathbb{C}$ be a smooth function of compact support. For $s \in \mathbb{C}$ let B_s denote one of J_s , Y_s or K_s . Then for $\alpha > 0$ and $j \in \mathbb{N}$ we have

$$(A.8) \quad \int_0^\infty F(x) B_s(\alpha\sqrt{x}) dx \\ = \pm \left(\frac{2}{\alpha}\right)^j \int_0^\infty \frac{\partial^j}{\partial x^j} (F(x) x^{-s/2}) x^{\frac{s+j}{2}} B_{s+j}(\alpha\sqrt{x}) dx.$$

Lemma A.5. ([Ol], equations 10.6.7 and 10.29.5) For $k = 0, 1, 2, \dots$,

$$(A.9) \quad J_s^{(k)}(z) = \frac{1}{2^k} \sum_{n=0}^k (-1)^n \binom{k}{n} J_{s-k+2n}(z),$$

$$(A.10) \quad e^{s\pi i} K_s^{(k)}(z) = \frac{1}{2^k} \left(e^{(s-k)\pi i} K_{s-k}(z) + \binom{k}{1} e^{(s-k+2)\pi i} K_{s-k+2}(z) \right. \\ \left. + \binom{k}{2} e^{(s-k+4)\pi i} K_{s-k+4}(z) + \dots + e^{(s+k)\pi i} K_{s+k}(z) \right).$$

B Mellin transforms

Lemma B.1. ([Ob], 2.19, page 15) Let $\phi(x) = (b + ax)^{-v}$. Then for $0 < \Re z < v$

$$(B.1) \quad \int_0^\infty \phi(x) x^{z-1} dx = (b/a)^z b^{-v} \frac{\Gamma(z)\Gamma(v-z)}{\Gamma(v)}.$$

Lemma B.2. ([Ob], 2.20, page 16) Let $\Re v > -1$ and

$$\phi(x) = \begin{cases} (a-x)^v & \text{if } x < a \\ 0 & \text{if } x > a \end{cases}.$$

Then for $\Re z > 0$

$$(B.2) \quad \int_0^\infty \phi(x) x^{z-1} dx = a^{v+z} \frac{\Gamma(v+1)\Gamma(z)}{\Gamma(v+z+1)}.$$

Lemma B.3. ([Ob], 2.21, page 16) Let $\Re v > -1$ and

$$\phi(x) = \begin{cases} (x-a)^v & \text{if } x > a \\ 0 & \text{if } x < a \end{cases}.$$

Then for $\Re z < -\Re v$

$$(B.3) \quad \int_0^\infty \phi(x) x^{z-1} dx = a^{v+z} \frac{\Gamma(-v-z)\Gamma(v+1)}{\Gamma(1-z)}.$$

Lemma B.4. ([BE], p.21) For $x > 0$,

$$(B.4) \quad J_{k-1}(x) = \frac{1}{2} \frac{1}{2\pi i} \int_{(\sigma)} \left(\frac{x}{2}\right)^{-s} \frac{\Gamma(s/2 + k/2 - 1/2)}{\Gamma(-s/2 + k/2 + 1/2)} ds,$$

where $-k+1 < \sigma < 1$.

Changing variable $-s := k - 1 + 2z$, we obtain

Lemma B.5.

$$(B.5) \quad J_{k-1}(x) = -\frac{1}{2\pi i} \int_{(\sigma)} \frac{\pi}{\Gamma(1+z)\Gamma(k+z)\sin(\pi z)} \left(\frac{x}{2}\right)^{k-1+2z} dz,$$

where $-k/2 < \sigma < 0$.

Let

$$(B.6) \quad \gamma(u, v) := \frac{2^{2u-1}}{\pi} \Gamma(u+v-1/2) \Gamma(u-v+1/2).$$

Lemma B.6. (*[K], p. 89*)

$$(B.7) \quad \int_0^\infty k_0(x, v) x^{w-1} dx = \gamma(w/2, v) \cos(\pi w/2),$$

$$(B.8) \quad \int_0^\infty k_1(x, v) x^{w-1} dx = \gamma(w/2, v) \sin(\pi v).$$

Corollary B.7.

$$(B.9) \quad k_1(x, 1/2 + ir) = \frac{\sin(\pi(1/2 + ir))}{2\pi i} \int_{(0.7)} x^{-2\beta} \gamma(\beta, 1/2 + ir) 2d\beta,$$

$$(B.10) \quad k_0(x, 1/2 + ir) = \frac{1}{2\pi i} \int_{(*)} x^{-2\beta} \gamma(\beta, 1/2 + ir) \cos(\pi\beta) 2d\beta,$$

where the contour of integration $(*)$ can be taken as $\Re\beta = -1$ except the area $|\Im\beta| < 1$, where it crosses the real axis at $\Re\beta > 0$.